本科生毕业论文

题目：模型范畴中的若干话题

Topics in Model Categories

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毕业论文题目	模型范畴中的若干话题		
导师评语	黄楚昊的论文叙述了 Quillen equivalence 的概念，陈述并且证明了 simplicial sets 和拓扑空间两个范畴作为 model 范畴，有 Quillen equivalence. 论文写作清楚，行文严密，引用规范。这是一篇优秀的本科毕业论文。 导师签名：方博汉 2022 年 5 月 26 日		

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摘要

本篇文章的主要目标在于证明单纯集合范畴的同伦范畴和拓扑空间范畴的同伦范畴是等价的。首先, 我们定义一般的模型范畴并引入 Quillen 等价的概念。我们将 Quillen 等价定义为诱导了同伦范畴等价的 Quillen 伴随函子。然后，我们利用组合信息给出单纯集合的定义。令人惊讶的是，我们可以模仿拓扑空间定义单纯集合上的许多同伦概念。为了联系单纯集合和拓扑空间，我们利用余单纯拓扑空间（cosimplicial topological space）构造它们之间的一对 Quillen 伴随函子。这一伴随函子被称为几何实现（geometric realization）和单纯函子（singular functor）。最后，我们证明这一对 Quillen 伴随函子是 Quillen 等价，这说明单纯集合范畴和拓扑空间范畴拥有相同的同伦信息。

关键词：模型范畴，单纯集合，同伦理论

Topics in Model Categories

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ABSTRACT

The main goal of this article is to present a proof for the equivalence between the homotopy categories of simplicial sets and of topological spaces. First, we define model categories in general and introduce Quillen equivalences between them, which are Quillen adjunctions that induce equivalences between their homotopy categories. Then, we define simplicial sets, which only contain combinatorial information. Surprisingly, we can define a lot of homotopy concepts for simplicial sets which resemble their topological counterparts. To connect simplicial sets and topological spaces, we construct a Quillen adjunction between them using cosimplicial topological space, which are called the geometric realization and the singular functor. Finally, we prove that this adjunction is a Quillen equivalence, which gives an equivalence between the homotopy theory of simplicial sets and of topological spaces.

KEY WORDS: Model Category, Simplicial Set, Homotopy Category

目录

第一章　Introduction

Mathematicians have long been familiar with the power of topological homotopy theory in algebraic topology. So natural questions arise: can we extend it to other fields of mathematics? Does something like a general homotopy theory exist somewhere in the world? These questions lead us to model categories, of which we can study the homotopy theory. In particular, we can study the homotopy theory of simplicial sets, and it resembles the homotopy theory of topological spaces in many ways. For instance, we can define simplicial homotopy groups as the counterparts of topological homotopy groups. And similarly to Serre fibrations, we can define Kan fibrations, which also induce long exact sequences of simplicial homotopy groups. These parallels between them make us wonder whether they carry the same homotopy information. And the answer is yes by our main theorem in the last section, which tells us that the homotopy categories of simplicial sets and of topological spaces are equivalent.

Apart from this rather classical result, model category also leads us to higher category theory, which is a recently developed but promising theory that can be applied to many different fields of mathematics. However, higher category theory is rather complicate and tends to be quite demanding to understand. Studying model categories may serve as a stepping-stone for understanding it, especially for $(\infty, 1)$-categories.

第二章　Model categories

2.1　Model categories

Definition 1. *A model category* M *is a category* M *equipped with three classes of morphisms* $(\mathrm{W}, \mathrm{C}, \mathrm{F})$ *which satisfy the following conditions:*

(MC1) M *is complete and cocomplete.*

(MC2) W *satisfies the "two out of three" property, meaning that for any three morphisms* (f, g, h) *verifying* $h = g \circ f$*, if any two of them belong to* W*, then the remaining one also.*

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "f"'] & & Z \\
& Y \arrow[ur, "g"'] &
\end{tikzcd}$$

(MC3) W, C, F *are stable under retract, meaning that if any morphism* f *is a retract of* g*, with* g *belonging to* W*(or* C*, or* F*), then* f *belongs to the same class. Here* $f : X \to Y$ *is a retract of* $g : X' \to Y'$ *means that there exist* i, r, j *and* s *making the following diagram commutes:*

$$\begin{tikzcd}
X \arrow[r, "i"'] \arrow[rr, bend left, "id_X"] \arrow[d, "f"'] & X' \arrow[r, "r"'] \arrow[d, "g"] & X \arrow[d, "f"] \\
Y \arrow[r, "j"] \arrow[rr, bend right, "id_Y"'] & Y' \arrow[r, "s"] & Y'
\end{tikzcd}$$

(MC4) For any pair of morphisms $(i : A \to X, p : E \to B)$ *such that* $i \in \mathrm{C}, p \in \mathrm{F} \cap \mathrm{W}$ *or* $i \in \mathrm{C} \cap \mathrm{W}, p \in \mathrm{F}$*, i always has the left-lifting property with respect to p, or equivalently, p has the right-lifting property with respect to i. That is to say, if we have two maps* $f : A \to X$ *and* $g : B \to Y$ *making the outer square of the following diagram commutes, then there always exists a diagonal map* $\phi : B \to X$*, called the lifting of g, making the whole diagram commutes.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook, "i"'] & E \arrow[d, two heads, "p"] \\
X \arrow[r, "g"'] \arrow[ur, dotted, "\exists\phi"] & B
\end{tikzcd}$$

(MC5) Every morphism f in M *can be decomposed as $p \circ i$ for some acyclic fibration p and cofibration i. It can also be decomposed as $q \circ j$ for some fibration q and acyclic cofibration j.*

$$\begin{tikzcd}
& C_f \arrow[dr, two heads, "p", "\sim"'] & \\
X \arrow[ur, hook, "i"] \arrow[rr, "f"] \arrow[dr, hook, "j"', "\sim"] & & Y \\
& P_f \arrow[ur, two heads, "q"'] &
\end{tikzcd}$$

Remark 1. *In the following, we will call the morphisms in* W *weak equivalences, denoted by* $\xrightarrow{\sim}$*, the morphisms in* C *cofibrations, denoted by* $\hookrightarrow$*, and the morphisms in* F *fibrations, denoted by* $\twoheadrightarrow$*. We will use these notations through out the article.*

Remark 2. *The elements in* $\mathrm{C} \cap \mathrm{W}$ *will be called acyclic cofibrations, denoted by* $\overset{\sim}{\hookrightarrow}$*. Similarly, the elements in* $\mathrm{F} \cap \mathrm{W}$ *will be called acyclic fibrations, denoted by* $\overset{\sim}{\twoheadrightarrow}$*. They are also called "trivial cofibrations" and "trivial fibrations" in some references.*

Remark 3. *We call an object $A \in$* M *a cofibrant object if the map $\varnothing \to A$ is a cofibration. We also say that A is cofibrant. Similarly, we call an object $X \in$* M *a fibrant object if the map $X \to *$ is a fibration.*

Remark 4. *Let A be any object in* M*. If we apply $(MC5)$ to the map $\varnothing \to A$, then we will get a factorization $\varnothing \hookrightarrow QA \xrightarrow{\sim} A$. The object QA is a cofibrant object and is weak homotopic to A via $QA \xrightarrow{\sim} A$. Therefore, we get a functor Q :* $\mathrm{M} \to \mathrm{M}$ *called the cofibrant replacement functor. Similarly, we can define the fibrant replacement functor R :* $\mathrm{M} \to \mathrm{M}$ *by factoring $X \to *$ as $X \xrightarrow{\sim} RX \twoheadrightarrow *$ for any $X \in$* M.

Remark 5. *We use the symbol $f \perp g$ to mean that f has the left lifting property with respect to g, which is equivalent to g having right lifting property with respect to f. For a set of morphisms S, we use $S^{\perp}$ to denote the set of morphisms that have the right lifting property with respect to every morphism in S. Similarly, we define ${}^{\perp}S$ to be the set of morphisms that have left lifting property with respect to S.*

Under this notation, $(MC4)$ is equivalent to $\forall i \in \mathrm{C}, \forall p \in \mathrm{F} \cap \mathrm{W}$ or $\forall i \in \mathrm{C} \cap \mathrm{W}, \forall p \in \mathrm{F}$, we have $f \perp g$. Or equivalently, $\mathrm{C} \subseteq^{\perp} (\mathrm{F} \cap \mathrm{W})$ and $\mathrm{C} \cap \mathrm{W} \subseteq^{\perp} \mathrm{F}$. In fact, the following theorem tells us that these two inclusions are both equal.

Theorem 1. $\mathrm{C} =^{\perp} (\mathrm{F} \cap \mathrm{W})$ *and* $\mathrm{C} \cap \mathrm{W} =^{\perp} \mathrm{F}$.

Proof: We only prove that $\mathrm{C} = {}^{\perp}(\mathrm{F} \cap \mathrm{W})$, the proof of the other case is similar. Suppose that $i : A \to X$ has the left lifting property with respect to $\mathrm{F} \cap \mathrm{W}$, we prove that it is a cofibration. By $(MC5)$, we can factor it as $i = p \circ j$ where $j : A \hookrightarrow W$ is a cofibration and $p : W \overset{\sim}{\twoheadrightarrow} X$ is an acylic fibration. Then i has the left lifting property with respect to p and we get a map $r : X \to W$ making the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, hook, "j"] \arrow[d, "i"'] & W \arrow[d, two heads, "p", "\sim"'] \\
X \arrow[r, "id_X"'] \arrow[ur, dotted, "\exists r"] & X
\end{tikzcd}$$

Then r fits into the following commutative diagram, which shows that i is a retract of $j \in \mathrm{C}$ and hence $i \in \mathrm{C}$:

$$\begin{tikzcd}
A \arrow[r, "id_A"'] \arrow[d, "i"'] \arrow[rr, bend left, "id_A"] & A \arrow[r, "id_A"'] \arrow[d, hook, "j"] & A \arrow[d, "i"] \\
X \arrow[r, "r"] \arrow[rr, bend right, "id_X"'] & W \arrow[r, "p"] & X
\end{tikzcd}$$

□

Using this theorem, one can easily prove the following useful theorem. We omit the proof here.

Theorem 2. *Cofibrations and acyclic cofibrations are preserved under pushouts. Similarly, fibrations and acyclic fibrations are preserved under pullbacks.*

Definition 2 (Quillen adjunction)**.** *Let* M *and* N *be two model categories, and* $F : \mathrm{M} \to \mathrm{N}$ *and* $G : \mathrm{N} \to \mathrm{M}$ *be two functors between the underlying categories. Then* F *and* G *is a Quillen adjunction between* M *and* N *if they are adjoint and satisfy the following equivalent conditions:*

- F *preserves cofibrations and acyclic cofibrations.*
- G *preserves fibrations and acyclic fibrations.*
- F *preserves cofibrations and* G *preserves fibrations.*
- F *preserves acyclic cofibrations and* G *preserves acyclic fibrations.*

One can easily prove that these four conditions are indeed equivalent using theorem 1.

2.2 The homotopy category

Definition 3 (The homotopy category)**.** *Let* M *be a model category, then its homotopy category* $Ho(\mathrm{M}) \coloneqq \mathrm{M}[\mathrm{W}^{-1}]$ *is the localization of* M *at* W*, together with the localization functor* $\lambda : \mathrm{M} \to Ho(\mathrm{M})$*. That is to say, for any functor* $F : \mathrm{M} \to \mathrm{C}$*, where* C *is an arbitrary category. If* F *maps* W *to isomorphisms in* C*, then it factors uniquely through* $\lambda : \mathrm{M} \to Ho(\mathrm{M})$*.*

Definition 4 (Left homotopy)**.** *Let* M *be a model category, and let* $A, X \in \mathrm{M}$*. Let* $f, g : A \to X$ *be two morphisms. Then* f *is left homotopic to* g*, denoted by* $f \simeq_l g$*, if there exists* $C \in \mathrm{M}$ *and morphisms making the following diagram commutes:*

$$\begin{tikzcd}
A \sqcup A \arrow[r, hook, "{(i_0,i_1)}"] \arrow[rr, bend left, "{(id_A,id_A)}"] \arrow[dr, "{(f,g)}"'] & C \arrow[r, "q", "\sim"'] \arrow[d, "H"] & A \\
 & X &
\end{tikzcd}$$

where $(i_0, i_1) : A \sqcup A \hookrightarrow C$ *is a cofibration and* $q : C \xrightarrow{\sim} A$ *is a weak equivalence. Any such object* C *together with these two maps is called a cylinder object of* A*.*

Lemma 1. *When* A *is cofibrant, then in any cylinder object as above,* C *is also cofibrant and the maps* $i_0, i_1 : A \to C$ *are acyclic cofibrations.*

Proof: Since A is cofibrant and the maps $r_0, r_1 : A \to A \sqcup A$ are the pushouts of $\varnothing \hookrightarrow A \hookleftarrow \varnothing$, they are cofibrations. Therefore, $i_0 = (i_0, i_1) \circ r_0$ is also a cofibration. Moreover, since $q \circ i_0 = id_A$ and q is a weak equivalence, i_0 is also a weak equivalence by the "two out of three" property. Therefore, i_0 is an acyclic cofibration, and so is i_1. □

Theorem 3. *When* A *is cofibrant, left homotopy defines an equivalence relation on* $Hom(A, X)$*.*

Proof: Only the transitivity needs to be proved. Let $f, g, h : A \to X$ be three morphisms such that $f \simeq_l g$ and $g \simeq_l h$. Then there exist two cylinder objects C and C' of A making the following two diagrams commute:

$$\begin{tikzcd}
A \sqcup A \arrow[r, hook, "{(i_0,i_1)}"] \arrow[rr, bend left, "{(id_A,id_A)}"] \arrow[dr, "{(f,g)}"'] & C \arrow[r, "q", "\sim"'] \arrow[d, "H"] & A \\
 & X &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \sqcup A \arrow[r, hook, "{(i'_0,i'_1)}"] \arrow[rr, bend left, "{(id_A,id_A)}"] \arrow[dr, "{(g,h)}"'] & C' \arrow[r, "q'", "\sim"'] \arrow[d, "H'"] & A \\
 & X &
\end{tikzcd}$$

Now, to construct a new cylinder object C''', we first consider the following pushout diagram which defines C'':

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[d, "i'_0"'] & C \arrow[d, "j"] \arrow[ddr, bend left, "q"] & \\
C' \arrow[r, "j'"'] \arrow[drr, bend right, "q'"'] & C'' \arrow[dr, dotted, "q''"] & \\
 & & A
\end{tikzcd}$$

and we consider the morphisms defined by

$$A \sqcup A \xrightarrow{(j\circ i_0, j'\circ i_1)} C'' \xrightarrow{q'':=(q,q')} A.$$

By lemma 1, i_1 and i'_0 are acylic cofibrations. So j and j', being their image under pushout, are acylic cofibrations. Since $q'' \circ j = q$, q'' is a weak equivalence by the "two out of three" property. However, $(i''_0, i''_1) := (j \circ i_0, j' \circ i_1)$ is not necessarily a cofibration. So we decompose it as

$$\begin{tikzcd}
(i''_0, i''_1) : A \sqcup A \arrow[r, hook, "(i'''_0, i'''_1)"] & C''' \arrow[r, two heads, "\phi", "\sim"'] & C'' ,
\end{tikzcd}$$

and define $q''' : C''' \to A$ to be $q''' := q'' \circ \phi$. Finally, we construct $H'' := (H, H') : C'' \to X$ and define $H''' := H'' \circ \phi : C''' \to X$. Since $H''' \circ i'''_0 = H'' \circ \phi \circ i'''_0 = H'' \circ i''_0 = H'' \circ j \circ i_0 = H \circ i_0 = f$ and similarly $H''' \circ i_1 = h$, we get the following commutative diagram:

$$\begin{tikzcd}
A \sqcup A \arrow[r, hook, "(i'''_0, i'''_1)"] \arrow[rr, bend left, "(i''_0, i''_1)"] \arrow[dr, "(f,h)"'] & C''' \arrow[r, two heads, "\phi", "\sim"'] \arrow[d, "H'''"'] \arrow[rrr, bend left, "q'''", "\sim"'] & C'' \arrow[r, "q''", "\sim"'] \arrow[dl, "H''"] & A \\
 & X & &
\end{tikzcd}$$

Therefore, we have $f \simeq_l h$ and thus $\simeq_l$ defines an equivalence relation. □

Theorem 4. *If X is fibrant, then for any morphism $h : B \to A$, the induced map $h^* : Hom(A, X) \to Hom(B, X)$ preserves $\simeq_l$.*

Proof: Let $f, g : A \to X$ be left homotopic, then we have a cylinder object C of A and a morphism $H : C \to X$ which gives us the left homotopy $f \simeq_l g$. Now, we can factor the weak equivalence $C \xrightarrow{\sim} A$ as the composition of an acylic cofibration $\phi : C \overset{\sim}{\hookrightarrow} D$ and an acyclic fibration $D \overset{\sim}{\twoheadrightarrow} A$, and we get the following commutative diagram:

$$\begin{tikzcd}
A \sqcup A \arrow[r, hook] \arrow[dr, "{(f,g)}"'] & C \arrow[r, hook, "\phi", "\sim"'] \arrow[d, "H"] & D \arrow[r, two heads, "\sim"'] \arrow[dl, dotted, "\exists H'"] & A \\
& X & &
\end{tikzcd}$$

Since X is fibrant and ϕ is an acylic cofibration, H factors through ϕ as $H = H' \circ \phi$. Now, we work with the cylinder object D of A instead of X, and we take an arbitrary cylinder object D' of B.

$$\begin{tikzcd}
A \sqcup A \arrow[r, hook, "{(i_0,i_1)}"] & D \arrow[r, two heads, "q", "\sim"'] & A \\
B \sqcup B \arrow[r, hook, "{(j_0,j_1)}"] & D' \arrow[r, "p", "\sim"'] & B
\end{tikzcd}$$

We consider the following diagram:

$$\begin{tikzcd}
B \sqcup B \arrow[r, "{(i_0 \circ h, i_1 \circ h)}"] \arrow[d, hook, "{(j_0,j_1)}"'] & D \arrow[d, two heads, "q", "\sim"'] \\
D' \arrow[r, "h \circ p"'] \arrow[ur, dotted, "\exists r"] & A
\end{tikzcd}$$

Since $q \circ (i_0 \circ h, i_1 \circ h) = (h, h) = (h \circ p) \circ (j_0, j_1)$, the outer square commutes. Then the acyclic fibration q has right lifting property with respect to the cofibration (j_0, j_1), so we get a lifting $r : D' \to D$. From this commutative diagram, one can verify that the following diagram commutes:

$$\begin{tikzcd}
B \sqcup B \arrow[r, hook, "{(j_0,j_1)}"] \arrow[d, "h \sqcup h"'] \arrow[dr, dotted] & D' \arrow[r, "p", "\sim"'] \arrow[d, "r"] \arrow[dr, dotted] & B \arrow[d, "h"] \\
A \sqcup A \arrow[r, hook, "{(i_0,i_1)}"'] \arrow[dr, "{(f,g)}"'] & D \arrow[r, two heads, "\sim", "q"'] \arrow[d, "H'"] & A \\
& X &
\end{tikzcd}$$

If we omit the central row of the diagram above, then we get a left homotopy between $f \circ h$ and $g \circ h$. □

Theorem 5. *For any morphism $h : X \to Y$, the induced map $h_* : Hom(A, X) \to Hom(A, Y)$ preserves $\simeq_l$.*

Proof: Let $f, g : A \to X$ be left homotopic, then we have the solid part of the following diagram:

$$\begin{tikzcd}
A \sqcup A \arrow[r, hook] \arrow[dr, "{(f,g)}"] \arrow[ddr, dotted, "{(f\circ h, g\circ h)}"'] & C \arrow[r, "\sim"] \arrow[d] & A \\
& X \arrow[d, dotted, "h"] & \\
& Y &
\end{tikzcd}$$

By adding the dotted morphisms in the diagram above, we get a left homotopy between $f \circ h$ and $g \circ h$. □

Therefore, when A is cofibrant, $\simeq_l$ defines equivalence relations on $Hom(A, X)$ and $Hom(A, Y)$, and we get an induced map $h_* : Hom(A, X)/ \simeq_l \to Hom(A, Y)/ \simeq_l$. The following theorem gives us a sufficient condition for this map to be an isomorphism.

Theorem 6. *When A is cofibrant, and $h : X \to Y$ is an acylic fibration or a weak equivalence between fibrant objects, then the induced map $h_* : Hom(A, X)/ \simeq_l \to Hom(A, Y)/ \simeq_l$ is an isomorphism.*

Proof: Case 1. When h is an acylic fibration.

Suppose that $f, g : A \to X$ verify that $h_*(f) = h_*(g)$. Then $f \circ h \simeq_l g \circ h$, and we have the following diagram:

$$\begin{tikzcd}
A \sqcup A \arrow[r, hook, "{(i_0,i_1)}"] \arrow[dr, "{(f\circ h, g\circ h)}"'] & C \arrow[r, "q", "\sim"'] \arrow[d, "H"] & A \\
& Y &
\end{tikzcd}$$

Now, we consider the following diagram:

$$\begin{tikzcd}
A \sqcup A \arrow[r, "{(f,g)}"] \arrow[d, hook, "{(i_0,i_1)}"'] & X \arrow[d, two heads, "h", "\sim"'] \\
C \arrow[r, "H"'] \arrow[ur, dotted, "\exists G"] & Y
\end{tikzcd}$$

Since h is an acylic fibration and (i_0, i_1) is a cofibration, the map H can be lifted to $G : C \to X$. Because G satisfies $G \circ (i_0, i_1) = (G \circ i_0, G \circ i_1) = (f, g)$ by the diagram above, it gives rise to a left homotopy between f and g as shown by the diagram bellow:

$$\begin{tikzcd}
A \sqcup A \arrow[r, hook, "{(i_0,i_1)}"] \arrow[dr, "{(f,g)}"'] & C \arrow[r, "q", "\sim"'] \arrow[d, "G"] & A \\
& X &
\end{tikzcd}$$

Therefore, h_* is injective.

Since $h : X \to Y$ is an acylic fibration and A is cofibrant, every morphism from A to Y can be lifted to morphism from A to X. Therefore, the map $h_* : Hom(A, X) \to Hom(A, Y)$ is surjective, and hence the induced map h_* after taking quotients is also surjective.

Case 2. When h is a weak equivalence between fibrant objects.

We decompose $(id_X, h) : X \to X \times Y$ as the composition of an acyclic cofibration $i : X \overset{\sim}{\hookrightarrow} W$ and a fibration $\pi : W \twoheadrightarrow X \times Y$. Then we get the diagram below:

$$\begin{tikzcd}
& X \arrow[d, hook, "i", "\sim"'] \arrow[ddl, bend right, "id_X"', "\sim"] \arrow[ddr, bend left, "h", "\sim"'] & \\
& W \arrow[d, two heads, "\pi"] \arrow[dl, two heads, "\alpha", "\sim"'] \arrow[dr, two heads, "\beta"', "\sim"] & \\
X & X \times Y \arrow[l, two heads, "p_X"'] \arrow[r, two heads, "p_Y"'] & Y
\end{tikzcd}$$

where $\alpha = p_X \circ \pi$ and $\beta = p_Y \circ \pi$. Since id_X, i, h are weak equivalences, α and β are also weak equivalences by the "two out of three" property. Meanwhile, since p_X and p_Y are pullbacks of $X \twoheadrightarrow * \twoheadleftarrow Y$, they are fibrations, and hence so are α and β. So α and β are acyclic fibrations. Now, we consider the image of this diagram under the functor $Hom(A, -)$. By case 1, α_* and β_* are isomorphisms. Since $(id_X)_*$ is the identity, $i_* = (\alpha_*)^{-1}$ is also an isomorphism, and hence $h_* = \beta_* \circ i_*$ is an isomorphism. □

Similar to the notion of left homotopy, we can define right homotopy by considering the path objects as follows.

Definition 5 (Right homotopy)**.** *Let* M *be a model category, and let* $A, X \in$ M*. Let* $f, g : A \to X$ *be two morphisms. Then* f *is right homotopic to* g*, denoted by* $f \simeq_r g$*, if there exists* $P \in$ M *and morphisms making the following diagram commutes:*

$$\begin{tikzcd}
X \arrow[r, "i", "\sim"'] & P \arrow[r, two heads, "(p_0{,}p_1)"] & X \times X \\
& A \arrow[u, "H"] \arrow[ur, "(f{,}g)"'] &
\end{tikzcd}$$

where $i : X \overset{\sim}{\to} P$ *is a weak equivalence and* $(p_0, p_1) : P \twoheadrightarrow X \times X$ *is a fibration. Any such object* P *together with these two maps are called a path object of* X*.*

And we have the similar properties for right homotopy.

- When X is fibrant, $\simeq_r$ defines an equivalence relation on $Hom(A, X)$.
- When A is cofibrant, the post-composition $h_* : Hom(A, X) \to Hom(A, Y)$ preserves $\simeq_r$ for any morphism $h : X \to Y$.

- Pre-compositions preserve right homotopy. When X is fibrant, the induced map

  $$h^* : Hom(B,X)/\simeq_r \to Hom(A,X)/\simeq_r$$

  is an isomorphism for any acyclic fibration $h : A \to B$ or weak equivalence between cofibrant objects.

Now, for any two objects A and X, there are two relations (not necessarily equivalence relations) $\simeq_l$ and $\simeq_r$ on $Hom(A,X)$. We wonder whether these two relations are equivalent. And we have the following result.

Theorem 7. *Let $f, g : A \to X$ be two morphisms.*

- *If A is cofibrant, then left homotopy implies right homotopy.*
- *If X is fibrant, then right homotopy implies left homotopy.*

Proof: We only prove the case when A is cofibrant, the other case is dual to this.

Since $f \simeq_l g$, we have the following diagram:

$$\begin{tikzcd}
A \sqcup A \arrow[r, hook, "{(i_0,i_1)}"] \arrow[dr, "{(f,g)}"'] & C \arrow[r, "q", "\sim"'] \arrow[d, "H"] & A \\
 & X &
\end{tikzcd}$$

We take an arbitrary path object of X, denoted by:

$$\begin{tikzcd}
X \arrow[r, "j", "\sim"'] & P \arrow[r, two heads, "{(p_0,p_1)}"] & X \times X
\end{tikzcd}$$

Then we consider the following diagram:

$$\begin{tikzcd}
A \arrow[r, "j\circ f"] \arrow[d, hook, "i_0"', "\sim"] & P \arrow[d, two heads, "{(p_0,p_1)}"] \\
C \arrow[r, "{(f\circ q,H)}"'] \arrow[ur, dotted, "\exists\phi"] & X \times X
\end{tikzcd}$$

The outer square commutes because $(f \circ g, H) \circ i_0 = (f \circ q \circ i_0, H \circ i_0) = (f, f)$ and $(p_0, p_1) \circ (j \circ f) = (p_0 \circ j \circ f, p_1 \circ j \circ f) = (f, f)$. Since A is cofibrant, i_0 is an acyclic cofibration by lemma 1. Meanwhile, (p_0, p_1) is a fibration. So there exists a lifting ϕ. Now, we define $G \coloneqq \phi \circ i_1 : A \to P$, then it fits into the following diagram:

$$\begin{tikzcd}
X \arrow[r, "j", "\sim"'] & P \arrow[r, two heads, "(p_0{,}p_1)"] & X \times X \\
& A \arrow[u, "G"] \arrow[ur, "(f{,}g)"'] &
\end{tikzcd}$$

This diagram commutes because $(p_0, p_1) \circ G = (p_0 \circ G, p_1 \circ G) = (p_0 \circ \phi \circ i_1, p_1 \circ \phi \circ i_1) = (f \circ q \circ i_1, H \circ i_1) = (f, g)$. Therefore, we have $f \simeq_r g$. □

Remark 6. *In particular, when* A *is cofibrant and* X *is fibrant,* $\simeq_l$ *and* $\simeq_r$ *defines the same equivalence relation on* $Hom(A, X)$*. We call it the homotopy relation, and denote it by* $\simeq$*.*

Let M be a model category, we use M_c to denote the full subcategory of M containing all cofibrant objects. Similarly, we define M_f to be the full subcategory of fibrant objects, and M_{cf} to be the full subcategory of objects that are fibrant and cofibrant simultaneously. Then we have inclusions $\mathrm{M}_{cf} \hookrightarrow \mathrm{M}_c \hookrightarrow \mathrm{M}$ and $\mathrm{M}_{cf} \hookrightarrow \mathrm{M}_f \hookrightarrow \mathrm{M}$. The question is whether these inclusions induce equivalences on homotopy categories? The answer is yes by the following theorem.

Theorem 8. *Let* M *be a model category, then we have equivalences*

$$Ho(\mathrm{M}_{cf}) \xrightarrow{\sim} Ho(\mathrm{M}_c) \xrightarrow{\sim} Ho(\mathrm{M}).$$

Proof: We only prove that $Ho(\mathrm{M}_c) \hookrightarrow Ho(\mathrm{M})$ is an equivalence, the rest follows easily.

The fibrant replacement functor $Q : \mathrm{M} \to \mathrm{M}_c$ induces a functor $Ho(Q) : Ho(\mathrm{M}) \to Ho(\mathrm{M}_c)$. We prove that this functor is inverse to the inclusion functor $Ho(\mathrm{M}_c) \hookrightarrow Ho(\mathrm{M})$. This is easy to be seen since the morphism $QX \to X$ in the cofibrant replacement $\varnothing \hookrightarrow QX \xrightarrow{\sim} X$ is a weak equivalence for any $X \in \mathrm{M}$, which induces an isomorphism in the homotopy category. □

The theorem above tells us that $Ho(\mathrm{M}) \simeq Ho(\mathrm{M}_{cf})$. Now, inside category M_{cf}, every object is cofibrant and fibrant simultaneously, and hence the homotopy relation defines equivalence relations on their morphism spaces by remark 6. By passing from morphisms to their homotopy classes, we define the following category.

Definition 6. *The category* $\pi\mathrm{M}_{cf}$ *consists of following datum:*

- $Obj(\pi\mathrm{M}_{cf}) = Obj(\mathrm{M}_{cf})$
- $Hom_{\pi\mathrm{M}_{cf}}(A, X) = Hom_{\mathrm{M}}(A, X)/\simeq$*, for any* $A, X \in \mathrm{M}_{cf}$

We notice that there is a natural functor $\lambda' : \mathrm{M}_{cf} \to \pi\mathrm{M}_{cf}$. The next theorem tells us that $\pi\mathrm{M}_{cf}$ together with λ' is equal to the homotopy category $Ho(\mathrm{M}_{cf})$ together with $\lambda : \mathrm{M}_{cf} \to Ho(\mathrm{M}_{cf})$.

Theorem 9. *$\lambda' : \mathrm{M}_{cf} \to \pi\mathrm{M}_{cf}$ is the homotopy category $\lambda : \mathrm{M}_{cf} \to Ho(\mathrm{M}_{cf})$.*

Proof: Recall that $\lambda : \mathrm{M}_{cf} \to Ho(\mathrm{M}_{cf})$ is the localization of M_{cf} with respect to weak equivalences. So we only need to verify that $\lambda' : \mathrm{M}_{cf} \to \pi\mathrm{M}_{cf}$ satisfies the universal property of localization.

Let $F : \mathrm{M}_{cf} \to \mathrm{C}$ be a functor which maps weak equivalences in M_{cf} to isomorphisms in C. Then we claim that there exists a unique functor $G : \pi\mathrm{M}_{cf} \to \mathrm{C}$ such that $F = G \circ \lambda'$. It is easy to see that the map of G on objects must coincide with F. So the only thing remains is to prove that F induces a map on morphism spaces, *i.e.*, to prove that if $f \simeq g : A \to X$ in M_{cf}, then $F(f) = F(g)$.

For any cylinder object $A \sqcup A \hookrightarrow W \xrightarrow{\sim} A$, we can always factor the morphism $W \xrightarrow{\sim} A$ as $W \overset{\sim}{\hookrightarrow} W' \overset{\sim}{\twoheadrightarrow} A$. By replacing W by W', we may assume that the morphism $W \to A$ is an acyclic fibration. Since A is cofibrant, $A \sqcup A$ is cofibrant and hence $W \in \mathrm{M}_{cf}$. The (left) homotopy between f and g gives us the following diagram:

$$\begin{tikzcd}
A \sqcup A \arrow[r, hook, "{(i_0,i_1)}"] \arrow[rr, bend left, "{(id_A,id_A)}"] \arrow[dr, "{(f,g)}"'] & W \arrow[r, two heads, "q", "\sim"'] \arrow[d, "H"] & A \\
 & X &
\end{tikzcd}$$

Since q is a weak equivalence in M_{cf}, $F(q)$ is an isomorphism in C. So from $id_A = q \circ i_0 = q \circ i_1$ we know that $F(i_0) = F(q)^{-1} = F(i_1)$, and hence $F(f) = F(H \circ i_0) = F(H) \circ F(i_0) = F(H) \circ F(i_1) = F(H \circ i_1) = F(g)$. □

Therefore, we have $Hom_{Ho(\mathrm{M}_{cf})}(A, X) \cong Hom_{\mathrm{M}}(A, X)/\simeq$ for any cofibrant object A and fibrant object X. Combining this with the equivalence $Ho(\mathrm{M}) \simeq Ho(\mathrm{M}_{cf})$, we get the following theorem.

Theorem 10. *Let* M *be a model category and A, X be two objects of* M*. Then we have*

$$Hom_{Ho(\mathrm{M})}(A, X) = Hom_{\mathrm{M}}(QA, RX)/\simeq .$$

Here, QA is cofibrant since it is the cofibrant replacement of A. Similarly, RX is fibrant and hence $\simeq$ defines an equivalence relation on $Hom_{\mathrm{M}}(QA, RX)$.

2.3 Derived functors and Quillen equivalences

Definition 7 (Total derived functor)**.** *Let* M *be a model category and* C *be a small category. Let* $\lambda : \mathrm{M} \to Ho(\mathrm{M})$ *be the natural functor. Then a total left derived functor* $\mathbb{L}F$*, if it exists, is a functor from* $Ho(\mathrm{M})$ *to* C *with a natural transformation* $\varepsilon : \mathbb{L}F\circ\lambda \Rightarrow F$ *satisfying the following universal property: for any functor* $G : Ho(\mathrm{M}) \to C$ *and any natural transformation* $\eta : G\circ\lambda \Rightarrow F$*, there exists an unique natural transformation* $\mu : G \Rightarrow \mathbb{L}F$ *such that* $\eta = \varepsilon\circ(\mu\circ\lambda)$*.*

$$\begin{tikzcd}
\mathrm{M} \arrow[r, "\mathbb{L}F\circ\lambda"'] \arrow[r, bend left=50, "F"] \arrow[r, phantom, shift left=3, "\Uparrow\varepsilon"] \arrow[d, "\lambda"'] & C \\
Ho(\mathrm{M}) \arrow[ru, "\mathbb{L}F"'] &
\end{tikzcd}$$

We can define total right derived functor in a similar way.

So a natural question is: when does a functor admit a total derived functor? The following theorem gives a sufficient condition.

Theorem 11. *Let* $F : \mathrm{M} \to \mathrm{C}$ *be a functor, where* M *is a model category and* C *is a small category. Then the total left derive functor* $\mathbb{L}F$ *of* F *exists if* F *maps the weak equivalences between cofibrant objects in* M *to isomorphisms in* C*.*

Proof: To define $\mathbb{L}F$, we first define a functor $F' : M \to \mathrm{C}$ as follows:

- for any object X, $F'(X) = F(QX)$,
- for any morphism f, $F'(f) = F(Qf)$.

Here, $Q : \mathrm{M} \to \mathrm{M}$ is the cofibrant replacement functor. To induce a functor $Ho(\mathrm{M}) \to \mathrm{C}$ from F', we only need to check that F' maps weak equivalences to isomorphisms. This is true since for any morphism $f : X \to Y$ in M, QX and QY are cofibrant objects, and $Qf : QX \to QY$ is a weak equivalence between cofibrant objects. We denote the induced functor by $\mathbb{L}F : Ho(\mathrm{M}) \to \mathrm{C}$. To construct the natural transformation $\varepsilon : \mathbb{L}F \circ \lambda \Rightarrow F$, we take any object $A \in \mathrm{M}$ and define $\varepsilon_A := F(i_A) : \mathbb{L}F \circ \lambda(A) = F(QA) \to F(A)$, where $i_A : QA \to A$ is the morphism in the cofibrant replacement of A. We will prove that $\mathbb{L}F$ and ε defined above satisfy the universal property.

Let $G : Ho(\mathrm{M}) \to \mathrm{C}$ be a functor and $\eta : G \circ \lambda \Rightarrow F$ be a natural transformation. Suppose we have a natural transformation $\mu : G \Rightarrow \mathbb{L}F$. Then $\eta = \varepsilon \circ (\mu \circ \lambda)$ is equivalent $\eta_A = \varepsilon_A \circ \mu_A$ for any $A \in \mathrm{M}$. Applying the natural transformation η to the map $i_A : QA \to A$, we get the following commutative diagram:

$$\begin{tikzcd}
G(A) \arrow[r, "\eta_A"] \arrow[dr, dotted, "\mu_A"] & F(A) \\
G(QA) \arrow[u, "G(i_A)"] \arrow[r, "\eta_{QA}"'] & F(QA) \arrow[u, "F(i_A)=\varepsilon_A"']
\end{tikzcd}$$

Since i_A is a weak equivalence and G is a functor from $Ho(\mathrm{M})$, $G(i_A)$ is an isomorphism, and hence the only μ_A that makes the diagram commute is $\mu_A = \eta_{QA} \circ G(i_A)^{-1}$. Therefore, the required natural transformation μ exists and is unique. □

Before proving the existence of derived functors for Quillen adjunctions, we need the following lemma.

Lemma 2. *Let $F : \mathrm{M} \to \mathrm{N}$ be a functor between model categories. If F maps acyclic cofibrations between cofibrant objects in* M *to weak equivalences in* N*, then it maps weak equivalences between cofibrant objects in* M *to weak equivalences in* N.

Proof: Let X and Y be two cofibrant objects in M and $f : X \xrightarrow{\sim} Y$ be a weak equivalence between them. Consider the left diagram and its image under F:

$$\begin{tikzcd}
X \arrow[rr, hook, "i_X"] \arrow[drr, hook, dotted, "\sim"] \arrow[ddrr, "f"', "\sim"] & & X \sqcup Y \arrow[d, hook, "i"] & & Y \arrow[ll, hook', "i_Y"'] \arrow[dll, hook', dotted, "\sim"'] \arrow[ddll, "id_Y", "\sim"'] \\
& & W \arrow[d, two heads, "q", "\sim"'] & & \\
& & Y & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
F(X) \arrow[rr, hook, "i_{F(X)}"] \arrow[drr, dotted, "\sim"] \arrow[ddrr, "F(f)"'] & & F(X) \sqcup F(Y) \arrow[d, "F(i)"] & & F(Y) \arrow[ll, hook', "i_{F(Y)}"'] \arrow[dll, dotted, "\sim"'] \arrow[ddll, "id_{F(Y)}"] \\
& & F(W) \arrow[d, "F(q)"] & & \\
& & F(Y) & &
\end{tikzcd}$$

The vertical maps i and q is a decomposition of $(f, id_Y) : X \sqcup Y \to Y$ such that i is a cofibration and q is an acyclic fibration. Then $X \sqcup Y$ and W are both cofibrant. Since f, id_Y and q are weak equivalences, two dotted maps $i \circ i_X$ and $i \circ i_Y$ are both weak equivalences by the "two out of three" property. Moreover, since i, i_X and i_Y are cofibrations, both dotted maps are acyclic cofibrations. So by the assumption of the lemma, both dotted maps in the right diagram are weak equivalences. Then, since $id_{F(Y)}$ is a weak equivalence, so is $F(q)$. Finally, $F(f)$ being the composition of $F(q)$ and the dotted map, is a weak equivalence as well by the "two out of three" property again. □

Corollary 1. *Let $F : \mathrm{M} \rightleftarrows \mathrm{N} : G$ be a Quillen adjunction, then it induces an adjunction between homotopy categories $\mathbb{L}F : Ho(\mathrm{M}) \rightleftarrows Ho(\mathrm{N}) : \mathbb{R}G$.*

Proof: Let $\lambda_{\mathrm{M}} : \mathrm{M} \to Ho(\mathrm{M})$ and $\lambda_{\mathrm{N}} : \mathrm{N} \to Ho(\mathrm{N})$ be the natural functors.

$$\begin{tikzcd}
\mathrm{M} \arrow[r, "F"] \arrow[d, "\lambda_{\mathrm{M}}"'] \arrow[dr, "\lambda_{\mathrm{N}} \circ F"] & \mathrm{N} \arrow[d, "\lambda_{\mathrm{N}}"] \\
Ho(\mathrm{M}) \arrow[r, dotted, "\mathbb{L}F"'] & Ho(\mathrm{N})
\end{tikzcd}$$

Now we consider the functor $\lambda_{\mathrm{N}} \circ F : \mathrm{M} \to Ho(\mathrm{N})$. Since F preserves acyclic fibrations, in particular, it preserves acylic fibrations between fibrant objects. Therefore, by lemma 2, F preserves weak equivalences between cofibrant objects, and hence $\lambda_{\mathrm{N}} \circ F$ maps weak equivalences between cofibrant objects to isomorphisms in $Ho(\mathrm{N})$. Now, by theorem 11, we know that the total left derived functor $\mathbb{L}F$ exists. Similarly, the total right derived functor $\mathbb{R}G$ exists.

Now, we remain to prove that $\mathbb{L}F : Ho(\mathrm{M}) \rightleftarrows Ho(\mathrm{N}) : \mathbb{R}G$ are adjoint.

Let X and Y be two objects in M and N respectively, then by the construction in theorem 11, we have $\mathbb{L}F(X) = \lambda_{\mathrm{N}} F(QX)$ and $\mathbb{R}G(Y) = \lambda_{\mathrm{M}} G(RY)$. So we have

$$Hom_{Ho(\mathrm{N})}(\mathbb{L}F(X), Y) \cong Hom_{Ho(\mathrm{N})}(\lambda_{\mathrm{N}} F(QX), Y) \cong Hom_{\mathrm{N}}(F(QX), RY)/\simeq,$$

and

$$Hom_{Ho(\mathrm{M})}(X, \mathbb{R}G(Y)) \cong Hom_{Ho(\mathrm{M})}(X, \lambda_{\mathrm{M}} G(RY)) \cong Hom_{\mathrm{M}}(QX, G(RY))/\simeq .$$

Since $F \dashv G$ is a Quillen adjunction, we have

$$Hom_{\mathrm{N}}(F(QX), RY) \cong Hom_{\mathrm{M}}(QX, G(RY)).$$

And we only need to show that this isomorphism preserves homotopy. Assume that $f, g : F(QX) \to RY$ are homotopic, and $f', g' : QX \to G(RY)$ are the corresponding morphisms. Since f is homotopic to g, we have the following diagram:

$$\begin{tikzcd}
RY \arrow[r, "p", "\sim"'] & P \arrow[r, two heads, "\pi"] & RY \times RY \\
 & F(QX) \arrow[u, "H"] \arrow[ur, "{(f,g)}"'] &
\end{tikzcd}$$

Then, under the isomorphism, we have a dual diagram:

$$\begin{tikzcd}
G(RY) \arrow[r, "G(p)", "\sim"'] & G(P) \arrow[r, two heads, "G(\pi)"] & G(RY) \times G(RY) \\
 & QX \arrow[u, "H'"] \arrow[ur, "{(f',g')}"'] &
\end{tikzcd}$$

Here, we have $G(RY\times RY)\cong G(RY)\times G(RY)$ since G is a right adjoint and preserves limits. $G(p)$ is a weak equivalence since G preserves weak equivalences between fibrant objects. $G(\pi)$ is a fibration since G preserves fibrations. H' is the dual of H. Therefore, the diagram above gives a homotopy $f'\simeq g'$. □

Definition 8 (Quillen equivalence)**.** *Let* M *and* N *be two model categories, and* $F:\mathrm{M}\rightleftarrows\mathrm{N}:G$ *be a Quillen adjunction. Then it is a Quillen equivalence if the following equivalent conditions are satisfied:*

- *the derived adjunction* $\mathbb{L}F:Ho(\mathrm{M})\rightleftarrows Ho(\mathrm{N}):\mathbb{R}G$ *is an equivalence.*
- *for any cofibrant object* X *in* M *and fibrant object* Y *in* N, *weak equivalences are preserved under the isomorphism*

$$Hom_{\mathrm{N}}(F(X),Y)\cong Hom_{\mathrm{M}}(X,G(Y)).$$

- *for any cofibrant object* X *in* M *and fibrant object* Y *in* N, *the derived adjunction unit*

$$X\to GF(X)\to GRF(X)$$

  is a weak equivalence in M, *and the derived adjunction counit*

$$FQG(Y)\to FG(Y)\to Y$$

  is a weak equivalence in N.

The theorem below is a criterion for a Quillen adjunction to be a Quillen equivalence, which will be useful in our proof of the main theorem.

Theorem 12. *Let* $F:\mathrm{M}\rightleftarrows\mathrm{N}:G$ *be a Quillen adjunction between two model categories. Then it is a Quillen equivalence if it satisfies following two conditions:*

- F *reflects weak equivalences, i.e., for any morphism* $f\in\mathrm{M}$, $F(f)$ *is a weak equivalence in* N *if and only if* f *is a weak equivalence in* M,
- *for any fibrant object* $Y\in\mathrm{N}$, *the counit* $\varepsilon_Y:FG(Y)\to Y$ *is a weak equivalence in* N.

Proof: We prove the last equivalent condition in definition 8. Suppose that X is a cofibrant object in M and Y is a fibrant object in N. Let $p_X:X\xrightarrow{\sim}RX$ be a fibrant replacement of X, and $i_Y:QY\xrightarrow{\sim}Y$ be a cofibrant replacement of Y.

First, we prove that the derived counit of Y is a weak equivalence. The map $FQG(Y)\to FG(Y)=F(QG(Y)\to G(Y))$ is a weak equivalence since the cofibrant replacement $QG(Y)\to G(Y)$ is a weak equivalence and F reflects weak equivalences. Meanwhile, $FG(Y)\to Y$ is a weak equivalence by the second assumption of the theorem. Therefore, the derived counit of Y, being the composition of two weak equivalences, is a weak equivalence.

Second, we prove that the derived unit of X is a weak equivalence. Since F reflects weak equivalences, we only need to prove that the image of this derived unit under F is a weak equivalence. We fit it into the following commutative diagram:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\varepsilon_X)"] \arrow[dr, "\sim", "id_{F(X)}"'] & FGF(X) \arrow[r, "FG(p_{F(X)})"] \arrow[d, "\varepsilon_{F(X)}"] & FGRF(X) \arrow[d, "\varepsilon_{RF(X)}", "\sim"'] \\
 & F(X) \arrow[r, "\sim", "p_{F(X)}"'] & RF(X)
\end{tikzcd}$$

The left triangle commutes by the counit-unit equation of the adjunction $F \dashv G$. Since $RF(X)$ is a fibrant object, the counit $\varepsilon_{RF(X)}$ is a weak equivalence. The map $p_{F(X)}$ is also a weak equivalence since it is the fibrant replacement of $F(X)$. Therefore, the upper horizontal map must be a weak equivalence by the "two out of three" property. □

2.4 Cofibrantly generated model categories

Definition 9 (Saturated class of maps). *Let* C *be a bicomplete category and* S *be a set of morphisms in* C. *Then* S *is saturated if it is closed under pushouts, transfinite compositions, and retracts.*

Definition 10. *Let* C *be a bicomplete category and* S *be a set of morphisms in* C, *then we define*

- $cell(S)$ *as the set of relative cell complexes, i.e., the maps of the form* $f : X_0 \to X$ *where* $X = colim_{n\in\mathbb{N}} X_n$ *and* X_n *is obtained from* X_{n-1} *by the pushouts along the coproducts of elements in* S*:*

$$\begin{tikzcd}
\sqcup A_\alpha \arrow[r] \arrow[d, "\sqcup f_\alpha"'] & X_{n-1} \arrow[d] \\
\sqcup B_\alpha \arrow[r] & X_n
\end{tikzcd}$$

  where f_α *are elements in* S.
- $cof(S)$ *as the set of retracts of elements in* $cell(S)$.

Definition 11 (Cofibrantly generated model category). *A model category* $\mathrm{M} = (\mathrm{M}, \mathrm{C}, \mathrm{W}, \mathrm{F})$ *is cofibrantly generated if there are two sets of morphisms, one is* $\mathscr{I}$*, called generating cofibrations, and the other is* $\mathscr{J}$*, called generating acyclic cofibrations, such that*

- $\mathrm{F} = \mathscr{J}^{\perp}$
- $\mathrm{F} \cap \mathrm{W} = \mathscr{I}^{\perp}$
- *the sources of elements in* $\mathscr{I}$ *are small relative to* $cell(\mathscr{I})$
- *the sources of elements in* $\mathscr{J}$ *are small relative to* $cell(\mathscr{J})$

Remark 7. *The last two conditions above, combined with the other conditions, guarantee that* $\mathscr{I}$ *and* $\mathscr{J}$ *admit small object arguments, which tells us that the set of cofibrations* C *is equal to* $cof(\mathscr{I})$*, and is the smallest saturated class that contains* $\mathscr{I}$*. Similarly, the set of acyclic cofibrations* $\mathrm{C} \cap \mathrm{W}$*, is equal to* $cof(\mathscr{J})$*, and is the smallest saturated class that contains* $\mathscr{J}$*. For further reading on small object argument, one can consult section 10.5 of Hirschhorn's book*[1] *and theorem 2.1.14 of Hovey's book*[2]*.*

第三章　Simplicial sets

3.1 Simplicial sets

Definition 12 (Simplex category)**.** *The simplex category, denoted by Δ, consists of following datum.*

- $Obj(\Delta) = \{[n] = \{0, 1, 2, ..., n\} | n \in \mathbb{N}\}$
- $Hom_{\Delta}([n], [m]) = \{f : [n] \to [m] | f \text{ is non-decreasing}\}$

Then simplicial sets are defined as contravariant functors from Δ to the category of sets.

Definition 13 (Simplicial set)**.** *Let Set denote the category of sets. Then the category of simplicial sets, denoted by $\hat{\Delta}$, is defined by*

$$\hat{\Delta} = Func(\Delta, Set).$$

So every simplicial set $X : \Delta \to Set$ contains a set $X_n := X([n])$ for every $n \in \mathbb{N}$. And for every non-decreasing function $f : [n] \to [m]$, these is an induced map between sets $f^* : X_m \to X_n$. Moreover, a morphism between two simplicial sets X and Y is equal to a series of maps $\phi_n : X_n \to Y_n$ for every $n \in \mathbb{N}$ such that they commute with the induced maps f^* on X and Y for any non-decreasing map f.

In particular, we consider two special kinds of non-decreasing maps, called face maps and degeneracy maps.

Definition 14 (Face maps and degeneracy maps)**.** *Let $n \in \mathbb{N}^*$ and $i \in [n]$, we define:*

- *face map $\delta_i^n : [n-1] \to [n]$ to be the unique strictly increasing map whose image leaves out $i \in [n]$,*
- *degeneracy map $\sigma_i^n : [n+1] \to [n]$ to be the unique surjective non-decreasing map such that $\sigma_i^n(i) = \sigma_i^n(i+1) = i$.*

When the index n is clear, we often omit it and just write δ_i and σ_i.

One can easily show that every non-decreasing map f can be generated by the face maps and degeneracy maps subject to the simplicial relations which is defined as follows.

Definition 15 (Simplicial relations)**.**

$$
\begin{aligned}
&\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1} && \textit{if } i < j \\
&\sigma_j \circ \sigma_i = \sigma_{i-1} \circ \sigma_j && \textit{if } i \geq j \\
&\sigma_j \circ \delta_i = \delta_i \circ \sigma_{j-1} && \textit{if } i < j \\
&\sigma_j \circ \delta_i = id && \textit{if } i = j \textit{ or } j+1 \\
&\sigma_j \circ \delta_i = \delta_{i-1} \circ \sigma_j && \textit{if } i > j+1
\end{aligned}
$$

Therefore, for a simplicial set X, we only need to consider the induced maps $d^i \coloneqq \delta_i^* : X_n \to X_{n-1}$ and $s^i \coloneqq \sigma_i^* : X_n \to X_{n+1}$, then all the other morphisms are generated by them. Such d^i are called coface maps and s^i are called codegeneracy maps. And they satisfy the cosimplicial relations dual to the simplicial relations.

In this way, we can write down X as follows:

$$\begin{tikzcd}
X_0 \arrow[r, shift right=6, "s^0"'] & X_1 \arrow[l, shift right=4, "d^0"'] \arrow[l, "d^1"'] \arrow[r, shift right=6, "s^0"'] \arrow[r, shift right=12, "s^1"'] & X_2 \arrow[l, shift right=8, "d^0"'] \arrow[l, shift right=4, "d^1"'] \arrow[l, "d^2"'] & \cdots
\end{tikzcd}$$

Recall that a simplicial set is a contravariant functor from Δ to Set, so we can consider the representable functor $Hom(-,[n]) : \Delta^{op} \to Set$ for every $n \in \mathbb{N}$. This simplicial set is called standard n-simplex, and denoted by Δ^n. Then by the Yoneda lemma, we have

$$Hom_{\Delta}(\Delta^n, X) \cong X([n]) = X_n.$$

Under this isomorphism, we can identify a map from Δ^n to X with a n-simplex of X. We will use this identification without indication in the rest of the article.

Except from Δ^n, another kind of simplicial sets that will be important in the future is horn which is defined as follows.

Definition 16 (Horn)**.** *Let $k, n \in \mathbb{N}$ and $0 \leq k \leq n$. The k-th horn of Δ^n, denoted by Λ_k^n, is defined by $(\Lambda_k^n)_i = \{f : [i] \to [n]$ such that $\{0, 1, ..., \hat{k}, ..., n\} \nsubseteq imf\}$. Or equivalently, Λ_k^n is the subcomplex of $\partial\Delta^n$ generated by all the non-degenerate simplexes of $\partial\Delta^n$ except for the k-th face of it.*

Remark 8. *If we view Λ_k^n as gluing n pieces of Δ^{n-1} along Δ^{n-2}, then one can easily prove that a map $\phi : \Lambda_k^n \to X$ is equivalent to n pieces of $(n-1)$-simplexes of X, denoted by*

$(v_0, v_1, ..., \hat{v_k}, ..., v_n)$ such that $d_i(v_j) = d_{j-1}(v_i)$ for any $0 \leq i < j \leq n$ such that $i, j \neq k$. Here, the equations guarantee that $\{v_i\}$ can glue along there boundaries.

3.2 Cosimplicial objects and adjunctions

Definition 17 (Cosimplicial objects)**.** *Let* C *be a category, then a cosimplicial* C*-object is a covariant functor $X : \Delta \to$ C.*

Example 1. *Consider the covariant functor $Y : \Delta \to \hat{\Delta}$ defined by $Y([n]) = \Delta^n$, the standard n-simplex, and $Y(f) = f_* : \Delta^n \to \Delta^m$, the morphism induced by any map $f : [n] \to [m]$ in Δ. Then Y defines a cosimplicial simplicial set.*

Definition 18 (Object Category)**.** *Let X be a simplicial set, then we define its object category $E(X)$ as follows:*

- *$Obj(E(X)) = \coprod_{n \in \mathbb{N}} X_n$,*
- *$Hom_{E(X)}(x, y) = \{\phi \in \Delta | X(\phi^{op})(y) = x\}$.*

There is a natural functor π from $E(X)$ to Δ, which is defined by $\pi(x) = [n]$ for any $x \in X_n$ and $\pi(\phi) = \phi$ for any $\phi \in Hom_{E(X)}(x, y)$.

Then we can write X as the colimit of standard simplexes by the following theorem.

Theorem 13. *Let X be a simplicial set, then we have*

$$X \cong colim_{E(X)} Y\pi.$$

Proof: The theorem is equivalent to X being the initial object in the category of cocones over $Y\pi$. First, we realize X as a cocone over $Y\pi$. For any $\omega \in E(X)$, we have $\omega \in X_n$ for some n, and thus ω corresponds to a map $f_\omega : Y\pi(\omega) = \Delta^n \to X$. Then, X together with the maps $\{f_\omega | \omega \in E(X)\}$ is a cocone over $Y\pi$. Let Z be any cocone over $Y\pi$, together with the maps $g_\omega : Y\pi(\omega) = \Delta^n \to Z$. Then we define a map G from X to Z by sending $\omega \in X_n$ to the n-simplex of Z represented by g_ω. Then $G \circ f_\omega(01...n) = G(w) = g_\omega(01...n)$ and hence $G \circ f_\omega = g_\omega$ for any $\omega \in E(X)$. Therefore, G defines a map from the cocone X to cocone Z. Moreover, such map is unique by the construction. So X is the initial object in the category of cocones over $Y\pi$. □

Theorem 14. *Let* C *be an arbitrary category, then any pair of adjunction $F : \hat{\Delta} \rightleftarrows$ C $: G$ is equivalent to a cosimplicial* C*-object.*

Proof: From adjunctions to cosimplicial C-objects.

We define $S = F \circ Y : \Delta \to \mathrm{C}$, where $Y : \Delta \to \hat{\Delta}$ is defined in example 1. Then S defines the cosimplicial C-object corresponding to the adjunction $F \dashv G$. Now, for any $X \in \hat{\Delta}$, we have $X \cong colim_{E(X)} Y\pi$ by theorem 13. Since F is a left adjoint, it preserves colimits and we have

$$F(X) \cong F(colim_{E(X)} Y\pi) \cong colim_{E(X)} FY\pi = colim_{E(X)} S\pi.$$

From this, we can easily see that different adjunctions correspond to different cosimplicial C-objects.

From cosimplicial C-objects to adjunctions.

Let $S : \Delta \to \mathrm{C}$ be a cosimplicial C-object. Then, for any simplicial set X, $S\pi$ is a functor from $E(X)$ to C. And we define $F : \hat{\Delta} \to \mathrm{C}$ by $F(X) := colim_{E(X)} S\pi$. In the other direction, given any object $c \in \mathrm{C}$, we define a simplicial set $Hom_{\mathrm{C}}(S, c)$ by $Hom_{\mathrm{C}}(S, c)_n := Hom_{\mathrm{C}}(S_n, c)$. And any morphism $\phi : [n] \to [m] \in \Delta$ induces $S(\phi) : S_n \to S_m$, and hence induces $S(\phi)^* : Hom_{\mathrm{C}}(S_m, c) \to Hom_{\mathrm{C}}(S_n, c)$. Therefore, we define a functor $G : \mathrm{C} \to \hat{\Delta}$ by $G(c) := Hom_{\mathrm{C}}(S, c)$. Next, we verify that (F, G) is indeed an adjunction. For any $X \in \hat{\Delta}$ and $c \in \mathrm{C}$, we have:

$$\begin{aligned}
Hom_{\mathrm{C}}(F(X), c) &\cong Hom_{\mathrm{C}}(colim_{E(X)} S\pi, c)) \\
&\cong lim_{x \in E(X)} Hom_{\mathrm{C}}(S\pi(x), c) \\
&\cong lim_{x \in E(X)} Hom_{\mathrm{C}}(S_n, c) \\
&\cong lim_{x \in E(X)} (Hom_{\mathrm{C}}(S, c))_n \\
&\cong lim_{x \in E(X)} Hom_{\hat{\Delta}}(\Delta^n, Hom_{\mathrm{C}}(S, c)) \\
&\cong lim_{x \in E(X)} Hom_{\hat{\Delta}}(Y\pi(x), Hom_{\mathrm{C}}(S, c)) \\
&\cong Hom_{\hat{\Delta}}(colim_{E(X)} Y\pi, Hom_{\mathrm{C}}(S, c)) \\
&\cong Hom_{\hat{\Delta}}(X, G(c)).
\end{aligned}$$

Finally, one can easily verify that these two procedures described above are inverse to each other. □

Remark 9. *Once given S, we know that $F(\Delta^n) = S_n$. To find out $F(X)$ for a general simplicial set X, we can first write X as a colimit of standard simplexes. Then we replace these simplexes by corresponding S_n. Finally, we take the colimit to get $F(X)$. In the proof above, we take the colimit $X = colim_{E(X)} Y\pi$. In general, this colimit can be replaced by any other colimit as* *long as the components of the colimit are standard simplexes.*

Remark 10. *From the proof of theorem 14, we also show that if a functor $F : \hat{\Delta} \to \mathrm{C}$ preserves colimits, then it admits a right adjoint G. In fact, the functor F gives rise to a cosimplicial C-object $S = F \circ Y$. And we can construct $F' : \hat{\Delta} \rightleftarrows \mathrm{C} : G$ as in the second part of the proof. Then $F(\Delta^n) = S_n = F'(\Delta^n)$. Moreover, F and F' both preserve colimits. Since any simplicial set can be written as a colimit of standard simplexes, functors F and F' must coincide for any simplicial set. Therefore, $F = F'$ admits a right adjoint G which is obtained by $G(X) = Hom_{\mathrm{C}}(S, X)$.*

3.3 Geometric realization

Now, to construct adjoint functors between $\hat{\Delta}$ and Top, we only need to find a cosimplicial topological space. The definition bellow defines a cosimplicial topological space that corresponds to the geometric realization and the singular functor.

Definition 19. *We use $|\Delta^n|$ to denote the standard geometric n-simplex which is defined as the convex hull of $n+1$ points $P_0^n, P_1^n, ..., P_n^n$ of generic positions in $\mathbb{R}^n$. Then any morphism $f : [n] \to [m]$ in Δ induces a continuous map $f_* : |\Delta^n| \to |\Delta^m|$ by sending P_i^n to $P_{f(i)}^m$ for any $i \in [n]$ and then extending it linearly. In this way, we define a cosimplicial topological space $|\Delta^\bullet| : \Delta \to Top$.*

Remark 11. *The reason why we use $|\Delta^n|$ to denote the standard geometric n-simplex will be clear after the following definition of geometric realization, see remark 12.*

Definition 20 (Geometric realization and the singular functor)**.** *By theorem 14, the cosimplicial topological space $|\Delta^\bullet|$ corresponds to an adjunction, denoted by*

$$|-| : \hat{\Delta} \rightleftarrows Top : S_\bullet(-).$$

For any simplicial set X, the topological space $|X|$ is called the geometric realization of X. And for any topological space T, the simplicial set $S_\bullet(T)$ is called the singular simplicial set of T.

Remark 12. *Recall the construction in theorem 14, we know that the geometric realization of the standard n-simplex Δ^n is the standard geometric n-simplex, which explains the notation $|\Delta^n|$.*

Theorem 15. *Let X be a simplicial set, then its geometric realization $|X|$ is a CW-complex.*

Proof: We define the n-th skeleton $sk_n(X)$ of X to be the subcomplex of X generated by all the simplexes of X of dimension $\leq n$. Then $X = \cup sk_n(X)$ and $sk_n(X)$ can be constructed from $sk_{n-1}(X)$ as its pushout along $\partial\Delta^n \hookrightarrow \Delta^n$:

$$\begin{tikzcd}
\sqcup_{\omega\in NX_n}\partial\Delta^n \arrow[r] \arrow[d, hook] & sk_{n-1}(X) \arrow[d] \\
\sqcup_{\omega\in NX_n}\Delta^n \arrow[r] & sk_n(X)
\end{tikzcd}$$

where $NX_n \subseteq X_n$ is the set of non-degenerate n-simplexes of X, *i.e.*,

$$NX_n \coloneqq \{\omega \in X_n | \nexists \alpha \in X_{n-1} \text{ such that } \omega = s^i(\alpha) \text{ for some } 0 \leq i \leq n-1\}.$$

Since $|-|$ is a left adjoint, it preserves colimits. Therefore, we have $|X| = \cup|sk_n(X)|$, where $|sk_n(X)|$ is obtained from $|sk_{n-1}(X)|$ by attaching $|\Delta^n| \cong \mathbb{D}^n$ along its boundary $|\partial\Delta^n| \cong \partial\mathbb{D}^n$ as follows:

$$\begin{tikzcd}
\sqcup_{\omega\in NX_n}|\partial\Delta^n| \cong \sqcup_{\omega\in NX_n}\partial\mathbb{D}^n \arrow[r] \arrow[d, hook] & |sk_{n-1}(X)| \arrow[d] \\
\sqcup_{\omega\in NX_n}|\Delta^n| \cong \sqcup_{\omega\in NX_n}\mathbb{D}^n \arrow[r] & |sk_n(X)|
\end{tikzcd}$$

This gives $|X|$ a CW-structure. □

In particular, $|X|$ is a compactly generated Hausdorff space, and we can view $|-|$ as a functor from $\hat{\Delta}$ to $CGHaus$, the category of compactly generated Hausdorff spaces. Then it preserves finite limits according to Gabriel and Zisman's paper[3].

Theorem 16. *The geometric realization* $|-| : \hat{\Delta} \to CGHaus$ *preserves finite limits.*

The reason why we consider the category $CGHaus$ instead of Top is that we do not always have $|X \times Y| \cong |X| \times |Y|$ in general. Instead, we have $|X \times Y| \cong |X| \times_{Ke} |Y|$ using the Kelley space product, which gives a different topology on the set $|X| \times |Y|$ to the usual topology.

第四章　Quillen model structures

4.1　Quillen structure on simplicial sets

Definition 21 (Quillen model structure on $\hat{\Delta}$)**.** *The Quillen model structure on* $\hat{\Delta}$ *consists of following datum:*

- *Cofibrations are monomorphisms* $f : X \to Y$*, which are those* f *such that it is injective at every level, i.e.,* $f_n : X_n \to Y_n$ *is injective for every* $n \in \mathbb{N}$*.*
- *Fibrations are Kan-fibrations, which are those* $f : X \to Y$ *that have right lifting property with respect to all horn inclusions.*

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\Delta^n \arrow[r] \arrow[ur, dotted, "\exists"] & Y
\end{tikzcd}$$

- *Weak equivalences are weak homotopy equivalences, i.e., those* $f : X \to Y$ *such that its geometric realization* $|f| : |X| \to |Y|$ *is a weak homotopy equivalence of topological spaces.*

According to chapter I, section 11 of Goerss and Jardine's book[4], this model structure is cofibrantly generated, with generating cofibrations $\mathscr{I}$ and generating acyclic cofibrations $\mathscr{J}$ given by:

$$\mathscr{I} = \{\partial\Delta^n \hookrightarrow \Delta^n\}_{n\geq 0} \text{ and } \mathscr{J} = \{\Lambda^n_k \hookrightarrow \Delta^n\}_{0\leq k\leq n}.$$

Definition 22 (Pushout-product)**.** *Let* M *be a category that is bicomplete. Let* $f : X \to Y$ *and* $g : A \to B$ *be two morphisms in* M*. Then we have the following commutative diagram:*

$$\begin{tikzcd}
X \times A \arrow[r, "f\times id_A"] \arrow[d, "id_X\times g"'] & Y \times A \arrow[d, "id_Y\times g"] \\
X \times B \arrow[r, "f\times id_B"'] & Y \times B
\end{tikzcd}$$

Therefore, it induces a map $f \square g : X \times B \cup_{X\times A} Y \times A \to Y \times B$*. This map is called the pushout-product of* f *and* g*.*

Similarly, if M admits internal homs, one can define the pullback-power

$$f^{\square g} : X^B \to X^A \times_{Y^A} Y^B.$$

Recall that we use the symbol $f \perp g$ to mean that f has the left lifting property with respect to g. Then we have the following useful result.

Lemma 3. *Let* M *be a bicomplete category that admits internal homs. Let* f, i, g *be three morphisms in* M, *then we have* $f \square i \perp g$ *if and only if* $f \perp g^{\square i}$.

Proof: Let $f : X \to Y$, $i : A \to B$, and $g : Z \to W$. Then $f \square i \perp g$ means that all commutative diagrams of the following form have liftings:

$$\begin{tikzcd}
X \times B \cup_{X \times A} Y \times A \arrow[r, "(\alpha,\beta)"] \arrow[d, "f \square i"'] & Z \arrow[d, "g"] \\
Y \times B \arrow[r, "\gamma"'] \arrow[ur, dotted, "\exists \phi"'] & W
\end{tikzcd}$$

Here $\alpha : X \times B \to Z$ and $\beta : Y \times A \to Z$ coincide over $X \times A$. And α, β, γ make the outer square commutes. The diagram above corresponds to the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\tilde{\alpha}"] \arrow[d, "f"'] & Z^B \arrow[d, "g^{\square i}"] \\
Y \arrow[r, "(\tilde{\beta},\tilde{\gamma})"'] \arrow[ur, dotted, "\exists \tilde{\phi}"] & Z^A \times_{W^A} W^B
\end{tikzcd}$$

Here $\tilde{\alpha} : X \to Z^B$ is the corresponding map of $\alpha : X \times B \to Z$ under the isomorphism $Hom_{\mathrm{M}}(X \times B, Z) \cong Hom_{\mathrm{M}}(X, Z^B)$. Similarly, we can define $\tilde{\beta}$, $\tilde{\gamma}$ and $\tilde{\phi}$. Then $(\alpha, \beta, \gamma, \phi)$ makes the upper diagram commutes if and only if $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\phi})$ makes the lower diagram commutes. $\square$

The proofs of following two facts can be found in Najib Idrissi's lecture notes on homotopy theory[5]. We use them to deduce two useful corollaries.

Fact 1. *Let* f *be an acylic cofibration in* $\hat{\Delta}$ *and* i *be a cofibration. Then* $f \square i$ *is an acylic cofibration.*

Corollary 2. *Let* X *be a fibrant object in* $\hat{\Delta}$ *and* A *be an arbitrary simplicial set. Then* X^A *is a fibrant object.*

Proof: We take $f = X \to *$ and $i : \varnothing \hookrightarrow A$. Then f is a fibration and i is a cofibration. Let g be any acyclic cofibration. Then by the lemma above, $g \square i$ is still an acyclic cofibration. Therefore, we have $g \square i \perp f$, and hence $g \perp f^{\square i}$ by lemma 3. Since g is an arbitrary acyclic cofibration, $f^{\square i}$ is a fibration. One can easily compute that $f^{\square i} : X^A \to *$, thus X^A is a fibrant object. $\square$

Corollary 3. *Let X be a fibrant object in $\hat{\Delta}$, then the map $X^{\partial\Delta^1} \to X^{\Delta^1}$ induced by the inclusion $\partial\Delta^1 \hookrightarrow \Delta^1$ is a fibration.*

Proof: We take $f : X \to *$ and $i : \partial\Delta^1 \hookrightarrow \Delta^1$. Then f is a fibration and i is a cofibration. By the same argument as in corollary 2, we know that $f^{\square i} : X^{\partial\Delta^1} \to X^{\Delta^1}$ is a fibration. □

Fact 2. *Let f be a cofibration in $\hat{\Delta}$, then $f \square i_\epsilon$ is an acylic cofibration, where i_ϵ, $\epsilon \in \{0,1\}$, are two possible inclusions $\Delta^0 \hookrightarrow \Delta^1$.*

Corollary 4. *Let X be a fibrant object in $\hat{\Delta}$, then $X^{\Delta^1} \to X^{\Delta^0}$ is an acyclic fibration.*

Proof: We take $f : X \to *$, and $i = i_\epsilon : \Delta^0 \to \Delta^1$. Let g be any cofibration. Then by the lemma above, $g \square i$ is an acylic cofibration. Therefore, we have $g \square i \perp f$ and hence $g \perp f^{\square i}$ by lemma 3. Since g is an arbitrary cofibration, $f^{\square i}$ is an acylic fibration. We finish the proof by computing that $f^{\square i} : X^{\Delta^1} \to X^{\Delta^0}$. □

4.2 Products and internal homs

Definition 23 (Products of simplicial sets)**.** *Let X and Y be two simplicial sets. Then their product $X \times Y$, which is still a simplicial set, is defined by:*

- $(X \times Y)_n = X_n \times Y_n$,
- $d^i_{X\times Y}(x,y) = (d^i_X(x), d^i_Y(y))$ *and* $s^i_{X\times Y}(x,y) = (s^i_X(x), s^i_Y(y))$.

For any $Y \in \hat{\Delta}$, we can prove that the functor $- \times Y : \hat{\Delta} \to \hat{\Delta}$ preserves colimits. First, one can easily prove that the colimit of simplicial sets is level-wise, *i.e.*, $(colim X_\alpha)_n = colim(X_\alpha)_n$. Then, we have:

$$\begin{aligned}((colim X_\alpha) \times Y)_n &\cong (colim X_\alpha)_n \times Y_n \cong (colim(X_\alpha)_n) \times Y_n \\ &\cong colim((X_\alpha)_n \times Y_n) \cong colim((X_\alpha \times Y)_n) \cong (colim(X_\alpha \times Y))_n.\end{aligned}$$

The isomorphism connecting two rows is because the functor $- \times S : Set \to Set$ where S is a set preserves colimits since it is left adjoint to $Hom_{Set}(S, -) : Set \to Set$.

Therefore, by remark 10, the functor $- \times Y : \hat{\Delta} \to \hat{\Delta}$ admits a right adjoint, which is the internal hom in $\hat{\Delta}$ defined as follows.

Definition 24 (Internal homs)**.** *Let X and Y be two simplicial sets. Then their internal hom Y^X or $\mathscr{H}om(X,Y)$, which is still a simplicial set, is defined by:*

- $(Y^X)_n = Hom_{\hat{\Delta}}(X \times \Delta^n, Y)$,

- *for any* $f : [n] \to [m]$ *in* Δ, *it induces* $f_* : \Delta^n \to \Delta^m$, *and hence induces* $f^* : (Y^X)_m = Hom_{\hat{\Delta}}(X \times \Delta^m \to Y) \to Hom_{\hat{\Delta}}(X \times \Delta^n \to Y) = (Y^X)_n$.

Therefore, we have $Hom_{\hat{\Delta}}(X \times Y, Z) \cong Hom_{\hat{\Delta}}(X, Z^Y)$ for any simplicial sets X, Y and Z.

4.3 Quillen structure on topological spaces

Definition 25 (Quillen structure on Top)**.** *The Quillen structure on* Top *consists of following datum:*

- *Cofibrations are the retracts of relative cell complexes.*
- *Fibrations are the Serre fibrations, maps witch have the right lifting property with respect to the inclusion* $i_0 : \mathbb{D}^n \hookrightarrow \mathbb{D}^n \times I$ *that includes* $\mathbb{D}^n$ *as* $\mathbb{D}^n \times \{0\} \subseteq \mathbb{D}^n \times I$ *for every* $n \in \mathbb{N}$.
- *Weak equivalences are the weak homotopy equivalences.*

According to chapter II, section 3 of Quillen's book[6], this model structure is cofibrantly generated, with generating cofibrations $\mathscr{I}$ and generating acyclic cofibrations $\mathscr{J}$ given by:

$$\mathscr{I} = \{\partial\mathbb{D}^n \hookrightarrow \mathbb{D}^n\}_{n\geq 1} \text{ and } \mathscr{J} = \{\mathbb{D}^n \hookrightarrow \mathbb{D}^n \times I\}_{n\geq 0}.$$

4.4 Simplicial homotopy groups

Recall that we can define the left homotopy relation in any model category using cylinder objects. In particular, in the case of $\hat{\Delta}$, we can take the cylinder object of A to be $A \sqcup A = A \times \partial\Delta^1 \hookrightarrow A \times \Delta^1 \xrightarrow{\sim} A$, and we can define homotopy in $\hat{\Delta}$ as follows.

Definition 26 (Homotopy in $\hat{\Delta}$)**.** *Let* $f, g : A \to X$ *be two morphisms between simplicial sets, the* f *is (left) homotopic to* g, *denoted by* $f \simeq g$ *if there exists* $h : A \times \Delta^1 \to X$ *such that the following diagram commutes:*

$$\begin{tikzcd}
A = A \times \Delta^0 \arrow[r, hook, "i_0"] \arrow[dr, "f"'] & A \times \Delta^1 \arrow[d, "h"] & A \times \Delta^0 = A \arrow[l, hook', "i_1"'] \arrow[dl, "g"] \\
& A &
\end{tikzcd}$$

Here, i_0 *and* i_1 *include* A *as* $A \times \{0\}$ *and* $A \times \{1\}$ *in* $A \times \Delta^1$ *respectively.*

Remark 13. *This homotopy relation may fail to be an equivalence relation on* $Hom_{\hat{\Delta}}(A, X)$ *in general. For example, we consider two maps* $i_0, i_1 : \Delta^0 \to \Delta^n$ *mapping* Δ^0 *to the* 0*-vertex and*

1-vertex of Δ^n. Then we obviously have $i_0 \simeq i_1$, but we don't have $i_1 \simeq i_0$. A remedy for this problem is to require X to be fibrant, then $\simeq$ becomes an equivalence relation by the following theorem.

Theorem 17. *Let $A, X \in \hat{\Delta}$ and X be a fibrant object under Quillen structure, i.e., X is a Kan complex. Then the homotopy relation defined above is an equivalence relation on $Hom_{\hat{\Delta}}(A, X)$.*

Proof: This relation is obviously reflexive, so we only need to check that it is symmetric and transitive.

First, we prove that it is symmetric. Let $f, g : A \to X$ satisfy $f \simeq g$. Then there exists a morphism $\phi : A \times \Delta^1$ such that $\phi(a, 0) = f(a)$ and $\phi(a, 1) = g(a)$. We now have two morphisms from $A \times \Delta^1$ to X, one is ϕ and the other is $g \circ p : A \times \Delta^1 \to A \to X$ where $p : A \times \Delta^1 \to A$ is the projection on the first factor. Since $A \times \Lambda^2_2$ is the pushout of $A \times \Delta^1 \hookleftarrow A \hookrightarrow A \times \Delta^1$, where both inclusions identify A with $A \times \{1\} \subseteq A \times \Delta^1$, and $\phi(a, 1) = g(a) = g \circ p(a, 1)$, we construct a map $H = (g \circ p, \phi) : A \times \Lambda^2_2 \to X$.

$$\begin{tikzcd}
& 1(f) \arrow[dr, "\phi"] & \\
0(g) \arrow[ur, dotted, "h"] \arrow[rr, "g\circ p"'] & & 2(g)
\end{tikzcd}$$

It corresponds to a map $H' : \Lambda^2_2 \to X^A$. By corollary 2, X^A is a fibrant object. Therefore, we can lift H' to $G' : \Delta^2 \to X^A$, which corresponds to $G : A \times \Delta^2 \to X$. Let $i = id_A \times \delta_2 : A \times \Delta^1 \hookrightarrow A \times \Delta^2$ and $h = G \circ i : A \times \Delta^1 \to X$. Then $h(a, 0) = G(a, 0) = H(a, 0) = g \circ p(a, 0) = g(a)$ and $h(a, 1) = G(a, 1) = H(a, 1) = \phi(a, 0) = f(a)$. Therefore, h gives a homotopy $g \simeq f$.

Next, we prove that it is transitive. Let $f, g, h : A \to X$ be three morphisms such that $f \simeq g$ and $g \simeq h$. Let ϕ and $\phi' : A \times \Delta^1 \to X$ be the homotopies $f \simeq g$ and $g \simeq h$, *i.e.*, $\phi(a, 0) = f(a)$ and $\phi(a, 1) = \phi'(a, 0) = g(a)$ and $\phi'(a, 1) = h(a)$. Since $A \times \Lambda^2_1$ is the pushout of $A \times \Delta^1 \hookleftarrow A \hookrightarrow A \times \Delta^1$, we can construct a map $H = (\phi, \phi') : A \times \Lambda^2_1 \to X$.

$$\begin{tikzcd}
& 1(g) \arrow[dr, "\phi'"] & \\
0(f) \arrow[ur, "\phi"] \arrow[rr, dotted, "\phi''"'] & & 2(h)
\end{tikzcd}$$

Similar to the proof of symmetry, we can lift H to a map $G : A \times \Delta^2 \to X$. And we define $\phi'' \coloneqq G \circ (id_A \times \delta_1) : A \times \Delta^1 \hookrightarrow A \times \Delta^2 \to X$ where $\delta_1 : \Delta^1 \hookrightarrow \Delta^2$ sends Δ^1 to the 1-edge of Δ^2.

Then h satisfies $\phi''(a,0) = G(a,0) = H(a,0) = \phi(a,0) = f(a)$ and $\phi''(a,1) = G(a,2) = H(a,2) = \phi'(a,1) = h(a)$. Therefore, ϕ'' gives a homotopy $f \simeq h$. □

Similarly, we can define the relative homotopy for any pair of simplicial sets $B \subseteq A$. Two morphisms $f, g : A \to X$ that are equal over B are homotopic relative to B if there exists a homotopy $h : A \times \Delta^1 \to X$ such that $h|_{B\times\Delta^1}(b,t) = f(b) = g(b)$. And one can prove as before that this defines an equivalence relation.

Now we are ready to define simplicial homotopy groups for a fibrant object $X \in \hat{\Delta}$. Recall that for a pointed topological space (S, s), we define $\pi_n(S, s)$ to be the homotopy classes of continuous maps from $(\mathbb{D}^n, \partial\mathbb{D}^n)$ to (S, s). By replacing the n-disk $\mathbb{D}^n$ by the standard n-simplex Δ^n, we get the following definition of simplicial homotopy groups.

Definition 27 (Simplicial homotopy groups)**.** *Let $X \in \hat{\Delta}$ be a fibrant object, i.e., X is a Kan complex, and let x be a vertex of X. Then its homotopy groups are defined by:*

$$\pi_n(X, x) \coloneqq Hom((\Delta^n, \partial\Delta^n), (X, x))/\simeq$$

Remark 14. *Let $\alpha : \Delta^n \to X$ satisfying $\alpha(\partial\Delta^n) = x$ represents an element of $\pi_n(X, x)$. Then we can view α as a n-simplex of X, and the condition $\alpha(\partial\Delta^n)$ is equivalent to $\partial(\alpha) = (x, x, ..., x)$. We denote it by $\partial(\alpha) = x$ for simplicity.*

Similar to the concatenation which defines a product structure on topological homotopy groups, we can also define a product structure on simplicial homotopy groups $\pi_n(X, x)$ for $n \geq 1$, making it into a group when $n \geq 1$ and moreover an Abelian group when $n \geq 2$.

Definition 28 (Group structure)**.** *Let X be a simplicial set and $\alpha, \beta : \Delta^n \to X$ represent elements of $\pi_n(X, x)$, where $n \geq 1$. By remark 14, we can view α and β as faces of X satisfying $\partial\alpha = \partial\beta = x$. Then we can define a map $\phi : \Lambda^{n+1}_n \to X$ by $\phi = (x, x, ..., x, \alpha, -, \beta)$ as described in remark 8. Since X is fibrant, the map ϕ can be lifted to a map $\omega : \Delta^{n+1} \to X$, which represents an element in X_{n+1}. Then, we define the product of α and β in $\pi_n(X, x)$ as the homotopy class represented by $d_n\omega : \Delta^n \to X$.*

Theorem 18. *The definition above is well-defined.*

Proof: First, the map ϕ exists since one can easily verify that $d_i(v_j) = d_{j-1}(v_i)$ for any $0 \leq i < j \leq n+1$ and $i, j \neq n$. Here, $v_i = x$ for $0 \leq i \leq n-2$, $v_{n-1} = \alpha$, and $v_{n+1} = \beta$.

Next, we prove that $d_n\omega : \Delta^n \to X$ maps $\partial\Delta^n$ to x. This is true since ϕ maps all $(n-1)$-simplexes of Λ^{n+1}_n to x. Therefore, $d_n\omega$ indeed represents an element in $\pi_n(X, x)$.

Finally, we prove that the homotopy class of $d_n\omega$ does not depend on the choice of the representatives α, β, and ω. Suppose that $h_{n-1} : \Delta^n \times \Delta^1 \to X$ is a homotopy $\alpha \simeq \alpha'$, $h_{n+1} : \Delta^n \times \Delta^1 \to X$ is a homotopy $\beta \simeq \beta'$, ω satisfies $\partial\omega = (x, ..., x, \alpha, d_n\omega, \beta)$, and ω' satisfies $\partial\omega' = (x, ..., x, \alpha', d_n\omega', \beta')$. Then we can construct the following diagram:

$$\begin{tikzcd}
(\Delta^{n+1} \times \partial\Delta^1) \cup (\Lambda^{n+1}_n \times \Delta^1) \arrow[r, "{((\omega,\omega'),(x,...,x,h_{n-1},-,h_{n+1}))}"] \arrow[d, hook] & X \\
\Delta^{n+1} \times \Delta^1 \arrow[ur, dotted, "\phi"'] &
\end{tikzcd}$$

The horizontal map comes from two maps. One is

$$(\omega, \omega') : \Delta^{n+1} \times \partial\Delta^1 = \Delta^{n+1} \times \{0\} \sqcup \Delta^{n+1} \times \{1\} \to X,$$

and the other is

$$(x, ...x, h_{n-1}, -, h_{n+1}) : \Lambda^{n+1}_n \times \Delta^1 \to X.$$

The well-definedness of this map is due to the following reason. Over the intersection

$$(\Delta^{n+1} \times \partial\Delta^1) \cap (\Lambda^{n+1}_n \times \Delta^1) = \Lambda^{n+1}_n \times \partial\Delta^1 = \Lambda^{n+1}_n \times \{0\} \sqcup \Lambda^{n+1}_n \times \{1\},$$

we have

$$\begin{aligned}
(\omega, \omega')|_{\Lambda^{n+1}_n \times \{0\}} &= \omega|_{\Lambda^{n+1}_n} = (x, ..., x, \alpha, -, \beta), \\
(\omega, \omega')|_{\Lambda^{n+1}_n \times \{1\}} &= \omega'|_{\Lambda^{n+1}_n} = (x, ..., x, \alpha', -, \beta'),
\end{aligned}$$

while

$$\begin{aligned}
(x, ..., x, h_{n-1}, -, h_{n+1})|_{\Lambda^{n+1}_n \times \{0\}} &= (x, ..., x, \alpha, -, \beta), \\
(x, ..., x, h_{n-1}, -, h_{n+1})|_{\Lambda^{n+1}_n \times \{1\}} &= (x, ..., x, \alpha', -, \beta').
\end{aligned}$$

Therefore, the horizontal map in the diagram above exists. Since X is fibrant and the vertical inclusion is an acylic cofibration, it can be lifted to a map $\phi : \Delta^{n+1} \times \Delta^1 \to X$. Now, the composition

$$\Delta^n \times \Delta^1 \xhookrightarrow{\delta_n \times id} \Delta^{n+1} \times \Delta^1 \xrightarrow{\phi} X$$

defines a homotopy $d_n\omega \simeq d_n\omega'$ (rel $\partial\Delta^n$). □

As we have mentioned before, the product defined above makes $\pi_n(X, x)$ into a group when $n \geq 1$ and into an Abelian group when $n \geq 2$. The proof of this result is not difficult but tedious and irrelevant to the main topic of this article. So we recommend those readers who are looking for a proof to consult Goerss and Jardine's brilliant book "Simplicial Homotopy Theory"[4].

In the case of topological spaces, a well know and powerful result is the long exact sequence induced by any Serre fibration. We have a similar result in the case of simplicial sets.

Theorem 19. *Let $p : X \to Y$ be a fibration between fibrant objects. Let x be a vertex of X, and we denote $F = X \times_Y \{p(x)\}$ to be the fiber of p over $p(x)$. Then we have a long exact sequence:*

$$... \to \pi_n(F, x) \to \pi_n(X, x) \to \pi_n(Y, p(x)) \to \pi_{n-1}(F, x) \to ...$$

The boundary map $\partial : \pi_n(Y, p(x)) \to \pi_{n-1}(F, x)$ is defined as follows. Let $\alpha : \Delta^n \to Y$ represents an element in $\pi_n(Y, p(x))$, then we consider the following diagram:

$$\begin{tikzcd}
\Lambda^n_n \arrow[r, "x"] \arrow[d, hook, "\sim"'] & X \arrow[d, two heads, "p"] \\
\Delta^n \arrow[r, "\alpha"'] \arrow[ur, dotted, "\exists\omega"] & Y
\end{tikzcd}$$

Since p is a fibration and the inclusion $\Lambda^n_n \hookrightarrow \Delta^n$ is an acylic cofibration, we can lift α to $\omega : \Delta^n \to X$. Since $p \circ \omega = \alpha$ maps $\partial\Delta^n$ to $p(x)$, ω maps $\partial\Delta^n$ to the fiber F and $d_n\omega$ defines a map from Δ^{n-1} to F. Moreover, $d_n\omega$ maps $\partial\Delta^{n-1}$ to x since $\partial\Delta^{n-1} \subseteq \Lambda^n_n$ and Λ^n_n is mapped to x under ω. Therefore, $d_n\omega$ represents an element in $\pi_{n-1}(F, x)$, and we define $\partial[\alpha] = [d_n\omega]$. The next theorem shows that $\partial[\alpha]$ is well-defined.

Theorem 20. *The definition above does not depend on the representatives α and ω.*

Proof: Let $H : \Delta^n \times \Delta^1$ be a homotopy between α and α'. Let ω and ω' be the lifting of α and α' to X respectively. We consider the following diagram:

$$\begin{tikzcd}
(\Lambda^n_n \times \Delta^1) \cup (\Delta^n \times \partial\Delta^1) \arrow[r, "{(x,(\omega,\omega'))}"] \arrow[d, hook, "i"', "\sim"] & X \arrow[d, two heads, "p"] \\
\Delta^n \times \Delta^1 \arrow[r, "H"'] \arrow[ur, dotted, "G"'] & Y
\end{tikzcd}$$

The upper horizontal map is well-defined because $\omega|_{\Lambda^n_n} = \omega'|_{\Lambda^n_n} = x$. Since i is an acylic cofibration and p is a fibration, we can lift H to G. Now, the composition

$$\begin{tikzcd}
\Delta^{n-1} \times \Delta^1 \arrow[r, hook, "{(\delta_n, id)}"] & \Delta^n \times \Delta^1 \arrow[r, "G"] & X
\end{tikzcd}$$

actually lands in F since $p \circ G|_{\partial\Delta^n \times \Delta^1} = H|_{\partial\Delta^n \times \Delta^1} = p(x)$. And it gives the homotopy between $d_n\omega$ and $d_n\omega'$. □

Proof: [Proof of Theorem 19] See lemma 3.4.9 of Hovey's book[2]. □

Lemma 4. *Let X be a simplicial set. If the morphism $f : X \overset{\sim}{\twoheadrightarrow} *$ is an acylic fibration, then $\pi_n(X, x) = 0$ for any $n > 0$.*

Proof: Let $\alpha : \Delta^n \to X$ represents an element in $\pi_n(X, x)$ with $n > 0$. Then we want to construct a homotopy between α and the constant map x relative to $\partial\Delta^n$. We consider the following diagram:

$$\begin{tikzcd}
(\Delta^n \times \partial\Delta^1) \cup (\partial\Delta^n \times \Delta^1) \arrow[r, "((x,\alpha),x)"] \arrow[d, hook] & X \arrow[d, two heads, "\sim"] \\
\Delta^n \times \Delta^1 \arrow[r] \arrow[ur, dotted, "\exists\phi"'] & *
\end{tikzcd}$$

The horizontal map restricts to α on $\Delta^n \times \{0\}$ and sends the rest to x. This map is well-defined since α maps $\partial\Delta^n$ to x. Now the inclusion, being a cofibration in $\hat{\Delta}$, has the left lifting property with respect to the acylic fibration $X \to *$. Therefore, there exists $\phi : \Delta^n \times \Delta^1 \to X$ which gives a homotopy between α and the constant map x relative to $\partial\Delta^n$. □

Lemma 5. *Let $f : X \to Y$ be a map between simplicial sets. If f is a Kan fibration, then its geometric realization $|f|$ is a Serre fibration.*

Proof: See corollary 3.6.2 of Hovey's book[2]. □

Theorem 21. *Let X be a simplicial set which is fibrant in the Quillen structure, x be a vertex of X. Then we have $\pi_0(X) \cong \pi_0(|X|)$, and $\pi_n(X, x) \cong \pi_n(|X|, |x|)$ for any $n > 0$.*

Proof: We prove by induction on n.

When $n = 0$, for any $x, x' \in X_0$, they are homotopic when there exists $y \in X_1$ such that $d^0(y) = x_0$ and $d^1(y) = x_1$. Suppose that $X = X_0 \sqcup X_1 \sqcup ... \sqcup X_k$ and that each X_i can not be decomposed as the disjoint union of two subcomplexes. Then, for any two points x, x' inside the same X_i, they are connected by a path of 1-simplexes $(y_0, y_1, ..., y_r)$ such that $d^0(y_0) = x, d^1(y_i) = d^0(y_{i+1})$ for $0 \leq i < r$, and $d^1(y_r) = x'$. Then $d^0(y_i)$ and $d^1(y_i)$ are connected by $y_i \in X_1$, and hence $[x] = [d^0(y_0)] = [d^1(y_0)] = [d^0(y_1)] = [d^1(y_1)] = ... = [d^1(y_r)] = [x']$. Therefore, $\pi_0(X) \cong \{0, 1, ..., k\}$. Meanwhile, $|X| = |X_1| \sqcup |X_2| \sqcup ... \sqcup |X_k|$, and one can easily check that each $|X_i|$ is connected. Thus, we have $\pi_0(X) \cong \{0, 1, ..., k\} \cong \pi_0(|X|)$.

Now, suppose that $\pi_{n-1}(X, x) \cong \pi_{n-1}(|X|, |x|)$, we prove that $\pi_n(X, x) \cong \pi_n(|X|, |x|)$.

We first define the path space PX as the pullback of the following diagram:

$$\begin{tikzcd}
PX \arrow[r] \arrow[d, "\pi"'] & X^{\Delta^1} \arrow[d, "(d^0, d^1)"] \\
X \arrow[r, "(x, id_X)"'] & X^{\partial\Delta^1} = X \times X
\end{tikzcd}$$

Since the map $PX \to X \to \{x\}$ is the pullback of $X^{\Delta^1} \to X^{\partial\Delta^1} \to X^{\Delta^0}$ which is an acylic fibration by corollary 4, $PX \to \{x\}$ is also an acylic fibration. Therefore, we have $\pi_n(PX, x) = 0$ for any $n \geq 1$ by lemma 4. Now, we define the loop space $\Omega X = PX \times_X \{x\}$, *i.e.*, the fiber of the fibration $PX \to X$ over x. Then we have the following long exact sequence by theorem 19:

$$\ldots \to \pi_n(PX, x) \to \pi_n(X, x) \to \pi_{n-1}(\Omega X, x) \to \pi_{n-1}(PX, x) \to \ldots$$

Since $\pi_n(PX, x) = 0$ for any $n \geq 1$ and $\pi_0(PX) \cong \pi_0(X)$, we have $\pi_n(X) \cong \pi_{n-1}(\Omega X)$ for any $n > 0$. Meanwhile, since $\pi : PX \to X$ is a (Kan) fibration by corollary 3, $|\pi| : |PX| \to |X|$ is a Serre fibration by lemma 5. Since $|-|$ preserves finite limits by theorem 16, we have $|\Omega X| = |PX| \times_{|X|} |x|$. Again, this Serre fibration induces a long exact sequence of homotopy groups and we get $\pi_n(|X|, |x|) \cong \pi_{n-1}(|\Omega X|, |x|)$. Now, combining these two isomorphisms with our induction hypothesis, we prove the isomorphism for n: $\pi_n(X, x) \cong \pi_{n-1}(\Omega X, x) \cong \pi_{n-1}(|\Omega X|, |x|) \cong \pi_n(|X|, |x|)$. □

4.5 The Quillen equivalence

Theorem 22 (Main theorem)**.** *The adjunction* $|-| : \hat{\Delta} \rightleftarrows Top : S_\bullet$ *is a Quillen equivalence. Here,* $\hat{\Delta}$ *and* Top *are both equipped with Quillen model structures.*

Proof: First, by definition, we know that the geometric realization preserves cofibrations and acyclic cofibrations, so they are Quillen adjunction by definition.

Then, we use theorem 12 to prove that it is a Quillen equivalence. First, we know that $|-|$ reflects weak equivalences by definition. Then, we need to prove that the counit $\varepsilon_X : S_\bullet(X) \to X$ is a weak equivalence in $\hat{\Delta}$ for any fibrant object X in Top. In fact, we can prove this for any topological space X, not necessarily fibrant. That is to say, we want to prove that $\pi_n(|S_\bullet(X)|, |x|) \to \pi_n(X, x)$ is a bijection when $n = 0$ and a group isomorphism when $n > 0$ for any $X \in Top$ and $x \in X$.

Now, an important observation is that $S_\bullet(X)$ is fibrant for any $X \in Top$. This is because under the adjunction of $|-|$ and $S_\bullet(-)$, the diagram below on the left is equivalent to the diagram on the right, where the dotted morphism clearly exists since $|\Lambda_i^n|$ is a deformation retract of $|\Delta^n|$.

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d, hook] & S_\bullet(X) \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} |\Lambda_i^n| \arrow[r] \arrow[d, hook] & X \\ |\Delta^n| \arrow[ur, dotted] & \end{tikzcd}$$

Therefore, by applying theorem 21 to $S_\bullet(X)$ we get

$$\begin{aligned} \pi_n(|S_\bullet(X)|, |x|) &\cong \pi_n(S_\bullet(X), x) \\ &\cong Hom_{\hat{\Delta}}((\Delta^n, \partial\Delta^n), (S_\bullet(X), x))/\simeq \\ &\cong Hom_{Top}((|\Delta^n|, |\partial\Delta^n|), (X, x))/\simeq \\ &\cong \pi_n(X, x) \end{aligned}$$

Here, the isomorphism

$$Hom_{\hat{\Delta}}((\Delta^n, \partial\Delta^n), (S_\bullet(X), x)) \cong Hom_{Top}((|\Delta^n|, |\partial\Delta^n|), (X, x))$$

is due to the adjunction $|-|$ and $S_\bullet(-)$. And we only need to verify that homotopy is preserved under this isomorphism. This is obvious by the isomorphism

$$\begin{aligned} &Hom_{\hat{\Delta}}((\Delta^n \times \Delta^1, \partial\Delta^n \times \Delta^1), (S_\bullet(X), x)) \\ \cong &Hom_{Top}((|\Delta^n \times \Delta^1|, |\partial\Delta^n \times \Delta^1|), (X, x)) \\ \cong &Hom_{Top}((|\Delta^n| \times I, |\partial\Delta^n| \times I), (X, x)). \end{aligned}$$

□

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致谢

四年弹指一挥间，我也即将正式告别北大，告别我的大学生活。虽然已在异国他乡独自求学近一年，但每每念及在燕园的生活，我的嘴角仍不禁浮上一抹微笑。五年前的我，在父母的陪伴下迈入了这个令我憧憬多年的学府。虽然我之前已经多次造访北大，但这次的心境既不同于儿时参观旅游景点般的兴奋，也不同于高中时参加选拔考试的紧张。我漫步在校园中，傍晚的微风抚摸着我的脸庞。我看着校园里面来来往往的学生和老师，第一次真实地感受到自己成为了北大的一员。我既骄傲又忐忑，努力地在脑海中规划自己的大学生活，希望能既不辜负自己十年来的努力也不辜负北大对我的期待。现在的我早已忘记了当初规划的内容，也很难说这四年是否辜负了自己的努力或者北大的期待。但是逝去的时间已经逝去，而我在北大经历过的一切，不论是快乐还是痛苦，都成为了我生命中重要的一部分。这四年来，我的家人以及很多老师和同学都给予了我莫大的支持和帮助，请让我在此感谢他们。由于人数众多，因此可能有所疏漏，还请理解。

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