Quillen's Theorem in Complex Cobordism and its Applications

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0 Introduction

From the Thom spectrum MU which is constructed from the universal complex vector bundles, one can define the associated generalized homology theory MU_* and cohomology theory MU^*. In 1960, Milnor computed the complex cobordism ring $MU^*(pt)$ in his paper [9] using the Adams spectral sequence. This spectral sequence is used to compute the stable homotopy groups, and was introduced by Adams in his work [1] in 1958. However, this is a rather complicated and powerful tool and involves the refined structural results for the Steenrod algebra. So one may wonder whether there is a more elementary proof of Milnor's result. And the answer is yes. In 1971, Quillen gave a new proof of Milnor's result in his work [10]. This new proof is more elementary in the sense that no Steenrod algebra or Adams spectral sequence is involved. The key of Quillen's proof relies on two different ways of constructing cohomology operations on MU^* and a relation between them. Instead of proving Milnor's result directly, he proved a new theorem which is now called Quillen's Theorem, and is presented as Theorem 4.2.1 in our article. Using his new method, he proved that $MU^*(X)$, as a module over the subring C of $\pi_*(MU)$ which is generated by the coefficients of the formal group law over the complex cobordism ring, is generated by the elements of non-negative degree. Then, as an immediate corollary, he showed that C is equal to $\pi_*(MU)$. Combining this with Lazard's Theorem which gives the structure of the universal formal group law, he was able to prove the Milnor–Quillen theorem. Our goal of this article is to follow Quillen's argument and to present the proof of Quillen's Theorem in [10].

We will begin with the axiomatic definitions of generalized (co)homology theories. Then Brown's representability theorem tells us that every cohomology theory comes from an Ω-spectrum. Conversely, given a spectrum, we are able to construct the associated (co)homology theories. A large class of spectra, called Thom spectra, can be constructed from Thom spaces of vector bundles. In particular, we are interested in the Thom spectrum MU that is constructed from complex vector bundles. We denote the associated (co)homology theories by MU_* and MU^*. Then we are able to define the Conner–Floyd classes in MU^* that resembles the Chern classes in ordinary cohomology. In particular, we are able to define Euler classes of complex line bundles, which further leads to the definition of a formal group law on $\pi_*(MU)$. This is the content of Chapter 1.

Since Thom spectra are constructed from vector bundles, one may expect that the associated (co)homology theories admit geometric interpretations. This is indeed the case, and is well explained by the Thom–Pontryagin construction. In particular, we are able to view $MU_*(X)$ as the complex bordism group of X for any topological space X. And moreover, the coefficient ring $\pi_*(MU)$ can be identified with the complex cobordism ring. It is not surprising that we can generalize the Thom–Pontryagin construction to MU^* and also give a geometric interpretation to it. But we have to assume X to be a manifold in this case. Both the Thom–Pontryagin construction and its generalization to cohomology is discussed in detail in Chapter 2.

In Chapter 3, we will define two cohomological operations on MU^*. One is the Landweber–Novikov operations, which come from a characteristic class in the cohomology MU^* that resembles the Chern polynomial in the ordinary cohomology, and the other is the Steenrod operations, which are defined using permutation group actions. It turns out that these two operations are closely related to each other, and we can work out the relation between them through a fixpoint argument.

Finally, the main theorem (Quillen's Theorem) is stated and proved in Chapter 4. The first key ingredient of the proof is the relation between cohomological operations that we have developed in the previous chapter. And the second key ingredient is a technical lemma that can be viewed as a refinement in the finite dimensional cases of the Gysin sequence associated with the universal $\mathbb{Z}/k$-bundle $\mathbb{S}^\infty \to \mathbb{S}^\infty/(\mathbb{Z}/k)$. After the proof of Quillen's Theorem, we will combine it with Lazard's Theorem to compute the coefficient ring $\pi_*(MU)$ and prove the Milnor–Quillen Theorem.

1 Generalized (co)homology theories and spectra

In this section, we will give the definitions of generalized homology and cohomology theories. They are in many ways similar to the ordinary (co)homology theories and share many properties. However, a huge difference is that their values on a point are no longer concentrated only on degree 0. Therefore, the coefficient ring of a generalized (co)homology theory can be rather complicated and may contain interesting topological information. Using Brown's representability theorem, one finds out that there is a strong relation between generalized cohomology theories and spectra. So we will define what a spectrum is and introduce the stable homotopy category.

There is a huge class of spectra that is of particular interest, namely the Thom spectra. This class of spectra is constructed from the $(\mathfrak{B}, f)$-structures, which are used to impose special structures on vector bundles. Therefore, their associated (co)homology theories have geometric interpretations via the Thom–Pontryagin construction. In particular, we will be interested in the Thom spectrum MU associated with the stable complex structure of vector bundles, which gives rise to the complex (co)bordism theory. The detailed discussion of this relation will be postponed until Chapter 2, where we will give the Thom–Pontryagin construction for homology theories and generalize it to cohomology theories. Now, let's begin with the definition of generalized (co)homology theories.

1.1 Generalized (co)homology theories

We have two versions of generalized (co)homology theories. We consider either the relative version, defined for a pair of CW-complexes $A \subseteq X$, or the reduced version, which concerns pointed CW-complexes. After defining them respectively, we will point out that they are in fact equivalent. In the following, we will only define generalized cohomology theories, and homology theories can be defined similarly.

Definition 1.1.1 (Generalized cohomology theories (relative version))**.** *A generalized relative cohomology theory h^* is a contravariant functor from the category of CW-pairs to the category of graded abelian groups, together with a natural transformation $\delta : h^*(A, \varnothing) \to h^{*+1}(X, A)$ that satisfy the following properties:*

- *(homotopy axiom) If $f, g : (X, A) \to (Y, B)$ are two morphisms between two CW-pairs that are homotopic, then we have*

$$f^* = g^* : h^*(Y, B) \to h^*(X, A).$$

- *(exactness axiom) For any CW-pair (X, A), we have a functorial long exact sequence*

$$\cdots \to h^*(X, A) \to h^*(X) \to h^*(A) \xrightarrow{\delta} h^{*+1}(X, A) \to \cdots$$

  where we use $h^*(Y)$ to denote $h^*(Y, \varnothing)$ and the first two maps are induced by inclusions.

- *(excision axiom) Let (X, A) be a CW-pair, and U be a subcomplex of X that satisfies $\bar{U} \subseteq \mathring{A}$. Then we have the excision isomorphism*

$$h^*(X, A) \xrightarrow{\sim} h^*(X - U, A - U),$$

  where the map is induced by inclusion.

Remark 1.1.2. *A cohomology theory h^* defined above is said to be additive if it further satisfies*

$$h^*(\coprod_i X_i, \coprod_i A_i) \cong \prod_i h^*(X_i, A_i),$$

where (X_i, A_i) are CW-pairs and the isomorphism is induced by inclusions.

Remark 1.1.3. *By the exactness axiom and diagram chasing, one can prove the long exact sequence for a triple (X, A, B) where $B \subseteq A \subseteq X$:*

$$\cdots \to h^*(X, A) \to h^*(X, B) \to h^*(A, B) \to h^{*+1}(X, A) \to \cdots$$

Here, the first two maps are induced by inclusions, and the last map is the composition of

$$h^*(A, B) \to h^*(A) \xrightarrow{\delta} h^{*+1}(X, A).$$

Now we give the definition of a reduced cohomology theory.

Definition 1.1.4. *(Generalized cohomology theories (reduced version)) A generalized reduced cohomology theory k^* is a contravariant functor from the category of pointed CW-complexes to the category of graded abelian groups, together with functorial suspension isomorphisms $\sigma : k^{*+1}(\Sigma X) \xrightarrow{\sim} k^*(X)$ that satisfy the following properties:*

- *(homotopy axiom) If $f, g : X \to Y$ are two morphisms between two pointed CW-complexes that are homotopic, then we have*

$$f^* = g^* : k^*(Y) \to k^*(X).$$

- *(exactness axiom) If (X, A) is a pointed CW-pair, then we have the following exact sequence*

$$k^*(X \cup_A CA) \to k^*(X) \to k^*(A),$$

where $X \cup_A CA$ is the mapping cone of the inclusion $A \hookrightarrow X$ and the maps are induced by inclusions.

Remark 1.1.5. *A reduced cohomology theory k^* defined above is said to satisfy the wedge axiom if it satisfies*

$$k^*(\bigvee_i X_i) \cong \prod_i k^*(X_i),$$

where X_i are pointed CW-complexes and the isomorphism is induced by inclusions.

As we have said at the beginning of this section, these two ways of defining a cohomology theory are actually the same, and their relation is given by the following.

Let h^* be a relative cohomology theory, then we define its associated reduced cohomology theory $K(h)^*$ to be

$$K(h)^*(X) = h^*(X, x_0),$$

where X is any pointed CW-complex, and $x_0 \in X$ is the basepoint. The suspension isomorphism $\sigma : K(h)^{*+1}(\Sigma X) \xrightarrow{\sim} K(h)^*(X)$ is given by

$$\begin{aligned}\sigma : K(h)^{*+1}(\Sigma X) = h^{*+1}(\Sigma X, pt) &\cong h^{*+1}(\Sigma X, C_- X) \\ &\cong h^{*+1}(C_+ X, X) \\ &\cong h^*(X, pt) = K(h)^*(X),\end{aligned}$$

where the first isomorphism comes from the homotopy axiom, the second from the excision axiom, and the last from the long exact sequence associated to the triple (C_+X, X, pt) which is explained in Remark 1.1.3.

Now one can prove that $K(h)^*$ really defines a reduced cohomology theory. The verification of the axioms is basically diagram chasing and is rather tedious, so we will omit it here.

In the opposite direction, given a reduced cohomology theory k^*, we can also associate a relative cohomology theory $H(k)^*$ to it, which is defined by

$$H(k)^*(X, A) = k^*(X \cup_A CA).$$

Here, $X \cup_A CA$ is a pointed CW-complex with the canonical basepoint. In particular, since $A \cup_{\varnothing} C\varnothing = A \sqcup \{pt\} = A_+$, we have $H(k)^*(A) = k^*(A_+)$. And the coboundary map $\delta : H(k)^*(A) \to H(k)^{*+1}(X, A)$ is given by

$$\delta : k^*(A_+) \cong k^{*+1}(\Sigma A_+) \to k^{*+1}(\Sigma A) \to k^{*+1}(X \cup_A CA) = H(k)^{*+1}(X, A),$$

where the first isomorphism is the suspension isomorphism of k^*, and the second morphism comes from the map $\Sigma A \to \Sigma A_+$ that collapses the two ends of ΣA, and the third morphism is induced by the map $X \cup_A CA \to \Sigma A$ where X is collapsed to a point.

Again, one may verify that $H(k)^*$ defined above really is a relative cohomology theory. Moreover, we have $H(K(h^*)) = h^*$ and $K(H(k^*)) = k^*$ for any relative cohomology h^* and reduced cohomology k^*. Therefore, these two definitions of a generalized cohomology theory are equivalent.

Remark 1.1.6. *One sees easily that additive relative cohomology theories correspond to reduced cohomology theories satisfying the wedge axiom.*

Remark 1.1.7. *Like the long exact sequence associated with a triple that is explained in Remark 1.1.3, the Mayer–Vietoris sequence can also be deduced from the axioms, and hence is satisfied by all generalized cohomologies. Again, the proof is just diagram chasing, so we will omit it here.*

1.2 Spectra

What we have done above is to define generalized (co)homologies abstractly using axioms. In fact, we can construct them concretely using spectra. And this begins with Brown's representability theorem which tells us when a contravariant functor from the category of pointed CW-complexes to abelian groups is representable.

Theorem 1.2.1 (Brown's representability theorem)**.** *Let F be a contravariant functor from the category of pointed CW-complexes to abelian groups. Assume that F satisfies the following requirements:*

- *(wedge axiom) Let X_i, $i \in I$ be pointed CW-complexes, then we have*

  $$F(\bigvee_i X_i) \cong \prod_i F(X_i),$$

  where the isomorphism is induced by inclusions .

- *(Mayer–Vietoris axiom) Let X be a CW-complex, and A, B be two subcomplexes of X such that $X = \mathring{A} \cup \mathring{B}$. Then we have the following exact sequence*

  $$F(X) \to F(A) \oplus F(B) \to F(A \cap B),$$

  where the first map is $F(i_A) \oplus F(i_B)$ and the second map is $F(j_A) - F(j_B)$. Here, $i_A : A \hookrightarrow X$ and $j_A : A \cap B \hookrightarrow A$ are inclusions and i_B, j_B are defined similarly.

Then there exists a pointed CW-complex K and an element $u \in F(K)$, such that the map $T_u(X) : [X, K] \to F(X)$ defined by $T_u(X)(f) = F(f)(u)$ is an isomorphism for any CW-complex X. Here, we use $[X, K]$ to denote the homotopy classes of continuous maps from X to K.

The proof of this theorem can be found in Brown's original paper [4] and will be omitted here.

Remark 1.2.2. *The CW-complex K is called the classifying space of the functor F, and u is the universal element.*

Remark 1.2.3. *In fact, in Brown's representability theorem, the way we associate a classifying space and an universal element to a functor is natural. To be specific, let G be another contravariant functor with associated classifying space L and universal element v. Then there is a bijection between the natural transformations from F to G and maps from K to L satisfying certain properties. Concretely, given a natural transformation $T : F \to G$, there exists a unique homotopy class of map $[f] \in [K, L]$ such that the following diagram commutes for any pointed CW-complex X*

$$\begin{array}{ccc} [X,K] & \xrightarrow{f_*} & [X,L] \\ \downarrow{\scriptstyle T_u(X)} & & \downarrow{\scriptstyle T_v(X)} \\ F(X) & \xrightarrow[T(X)]{} & G(X). \end{array}$$

In particular, by taking $X = K$ in the diagram above, we get the property that the map f should satisfy: $G(f)(v) = T(X)(u)$.

Let k^* be a reduced cohomology theory, then each k^n satisfies the requirements of Brown's representability theorem. Therefore, for every $n \in \mathbb{Z}$, there exists a pointed CW-complex K_n and an element $u_n \in k^n(K_n)$ such that the map

$$T_{u_n} : [X, K_n] \to k^n(X)$$

defined by the pullback of u_n is an isomorphism.

Now we focus on the suspension isomorphism $\sigma : k^{n+1}(\Sigma X) \xrightarrow{\sim} k^n(X)$. Since we have

$$k^{n+1}(\Sigma X) \cong [\Sigma X, K_{n+1}] \cong [X, \Omega K_{n+1}],$$

we may view $k^{n+1}(\Sigma X)$ as a functor represented by ΩK_{n+1}. Then by Remark 1.2.3, the isomorphism σ corresponds to a homotopy equivalence ϕ_n from ΩK_{n+1} to K_n. This leads to the following definition of a spectrum.

Definition 1.2.4. *A spectrum E_* is a sequence of pointed CW-complexes E_n indexed by $n \in \mathbb{N}$ together with cellular structure maps $\alpha_n : \Sigma E_n \to E_{n+1}$.*

Remark 1.2.5. *Instead of considering the structure maps $\alpha_n : \Sigma E_n \to E_{n+1}$ themselves, we can also consider their adjoint maps $\beta_n : E_n \to \Omega E_{n+1}$. And they also play an important role in theories concerning spectra.*

Remark 1.2.6. *In some references, it is required that the structure maps α_n be inclusions of subcomplexes. This is not far from our definition, since we can always reconstruct our spectrum term by term using mapping cylinders. The resulting spectrum is homotopic to our original one with structure maps being inclusions of subcomplexes. This construction is called the telescope construction, and is well explained in Chapter 4, Part III of Adams' book [2].*

Therefore, using Brown's representability theorem, we have constructed a spectrum K_* from the reduced cohomology theory k^*. Moreover, the adjoint structure maps $\phi_n : K_n \to \Omega K_{n+1}$ of this spectrum are homotopy equivalences. We call a spectrum satisfying this condition an Ω-spectrum.

The next step is to define morphisms between spectra. Since each spectrum is a sequence of pointed CW-complexes and what we are really interested in is what happens when the index tends to infinity, we should allow a morphism to be defined on a given cell of E_n only after possibly increasing n by the structure maps. To this end, we quote Adams' slogan "cells now - maps later" from his book [2]. The precise definition of morphisms between spectra can be found in Chapter 2, Part III of the same book of Adams and will be omitted here. The category consists of spectra and morphisms between them is called the stable homotopy category. Following Adams' notation, we use $[E, F]_r$ to define the homotopy classes of morphisms of degree r from the spectrum E to F. Here, a morphism of degree r means that it decreases the indexes of the components of E by r. Or naively, it maps E_n to F_{n-r} for every $n \in \mathbb{N}$.

Apart from the spectra associated with cohomology theories, there are many ways to construct spectra. For example, let X be any pointed CW-complex, then we can construct a spectrum $\Sigma^\infty X$ from it by defining the n-th component to be $(\Sigma^\infty X)_n := \Sigma^n X$. And the structure maps are taken to be the identity maps $\Sigma(\Sigma^n X) = \Sigma^{n+1} X$. This spectrum is called the suspension spectrum of X. For every $n \in \mathbb{N}$, we use $\Sigma^{\infty - n} X$ to denote the original spectrum $\Sigma^\infty X$ shifted by n, *i.e.*, we have $(\Sigma^{\infty-n} X)_k = \Sigma^{k-n} X$ for $k \geqslant n$ and $(\Sigma^{\infty-n} X)_k = \Omega^{n-k} X$ for $k < n$. And the structure maps are natural. If we take our X to be $\mathbb{S}^0$, *i.e.*, the disjoint union of two points with any point being the basepoint, then the n-th component of its suspension spectrum is the n-dimensional sphere $\mathbb{S}^n$. This spectrum is called the sphere spectrum and is denoted by $\underline{\mathbb{S}}$. Recall that homotopy groups of a topological space are defined by considering the maps from spheres to it. Similarly, to define homotopy groups for a spectrum, we can consider the morphisms from the sphere spectrum to it.

Definition 1.2.7. *Let E be a spectrum, then its n-th homotopy group is define by $\pi_n(E) = [\underline{\mathbb{S}}, E]_n$.*

The following proposition gives a concrete description of these homotopy groups. The proof is straightforward once one knows the definition of morphisms between spectra.

Proposition 1.2.8. *Let X be a pointed CW-complex and E be a spectrum, then we have*

$$[\Sigma^\infty X, E]_n \cong \varinjlim [\Sigma^{n+k} X, E_k],$$

where the colimit is taken with respect to the following maps

$$[\Sigma^{n+k}X, E_k] \to [\Sigma^{n+k+1}X, \Sigma E_k] \to [\Sigma^{n+k+1}X, E_{k+1}].$$

Here, the first map is induced by taking suspension, and the second is induced by the structure maps of E. In particular, we can take X to be $\mathbb{S}^0$, then its suspension spectrum is $\Sigma^\infty X = \underline{\mathbb{S}}$ and we get

$$\pi_n(E) = [\underline{\mathbb{S}}, E]_n \cong \varinjlim [\mathbb{S}^{n+k}, E_k] = \varinjlim \pi_{n+k}(E_k).$$

However, unlike the case of topological spaces where the homotopy groups are only defined for $n \in \mathbb{N}$, the homotopy groups of a spectrum can be defined for any $n \in \mathbb{Z}$. Moreover, $\pi_n(E)$ are often non trivial even for $n < 0$. In fact, the spectrum E satisfying $\pi_n(E) = 0$ for any $n < 0$ is called a connective spectrum.

Apart from taking iterated suspension of a pointed CW-complex, another way to construct spectra is highly related to the special structures imposed on vector bundles, and the resulting spectra are called Thom spectra. In our case, we are interested in stable complex structures on vector bundles and we recall the construction of the corresponding Thom spectrum MU.

For every $n \in \mathbb{N}$, we have the universal complex vector bundle of rank n, denoted by $EU(n) \to BU(n)$. We denote by $MU(n)$ the Thom space $T(EU(n))$ of this vector bundle which is defined by collapsing the boundary of the associated disc bundle to a point. Now we take the direct sum of the bundle $EU(n)$ with a trivial complex line bundle over $BU(n)$, and get $EU(n) \oplus \mathbb{C}$ over $BU(n)$. It has rank $n+1$ and hence is classified by some map $BU(n) \to BU(n+1)$ as follows

$$\begin{array}{ccc} EU(n) \oplus \mathbb{C} & \longrightarrow & EU(n+1) \\ \downarrow & & \downarrow \\ BU(n) & \longrightarrow & BU(n+1). \end{array}$$

It induces a map between Thom spaces $T(EU(n) \oplus \mathbb{C}) \to MU(n+1)$. Since we have $T(EU(n) \oplus \mathbb{C}) \cong \Sigma^2 MU(n)$, we get a map $\Sigma^2 MU(n) \to MU(n+1)$. Using these as structure maps, we can define a spectrum MU by defining its $2n$-th component MU_{2n} to be $MU(n)$ and its $(2n+1)$-th component MU_{2n+1} to be $\Sigma MU(n)$.

It is not surprising that spectra resemble CW-complexes in many ways, and often have better homotopic behavior. Many results that are only valid in the stable range in the context of CW-complexes will be always true in the context of spectra. This is due to the reason that we only care about the phenomenon at "infinity" in spectra. First, we recall that Whitehead theorem tells us that a weak homotopy equivalence between CW-complexes is always a homotopy equivalence. And this still holds for spectra.

Proposition 1.2.9. *Let $f : E \to F$ be a map between spectra such that the induced map $f_* : \pi_n(E) \to \pi_n(F)$ is an isomorphism for any n. Then f is an isomorphism in the stable homotopy category.*

The following proposition implies that we do somehow obtain stability by passing from CW-complexes to spectra.

Proposition 1.2.10. *Let E and F be two spectra, then we have*

$$[E, F]_* \cong [\Sigma E, \Sigma F]_*,$$

where the isomorphism is induced by taking suspension. In particular, we can take E to be $\mathbb{S}$, and get $\pi_*(E) \cong \pi_{*+1}(\Sigma E)$.

Both propositions above are proved in Chaoter 3, Part III of Adams' book [2]. For those who are interested in other properties of spectra, the same chapter of Adams' book can be used as a great reference.

Remark 1.2.11. *Using the morphism $\Sigma E \to \Sigma E \vee \Sigma E$ which collides $E \subseteq \Sigma E$, we can equip the set $[\Sigma E, \Sigma F]$ with a group structure. This group structure can be pulled back to $[E, F]$ via the isomorphism $[E, F] \cong [\Sigma E, \Sigma F]$. Moreover, using the isomorphism $[E, F] \cong [\Sigma E, \Sigma F] \cong [\Sigma^2 E, \Sigma^2 F]$, this group structure can be proved to be abelian. And one may verify that composition is bilinear. Therefore, the homotopy category is an additive category once we can show the existence of finite coproducts. In fact, we can do better by showing that there exists arbitrary coproducts and products. And they are defined by taking termwise wedges and products respectively. A full discussion of this result can be found also in Chapter 3, Part III of Adams' book [2].*

Next, we will construct generalized (co)homology theories from the spectrum MU. In fact, this construction can be done to any spectrum.

Definition 1.2.12. *Let E_* be a spectrum, then we define its associated reduced (co)homology theories as follows*

$$\widetilde{E}_n(X) = \pi_n(E \wedge X),$$
$$\widetilde{E}^n(X) = [\Sigma^\infty X, E]_{-n},$$

where X is any pointed CW-complex. The suspension isomorphisms are given by

$$\widetilde{E}_n(X) = \pi_n(E \wedge X) \cong \pi_{n+1}(\Sigma(E \wedge X)) = \pi_{n+1}(E \wedge \Sigma X) = \widetilde{E}_{n+1}(\Sigma X),$$

and

$$\widetilde{E}^{n+1}(\Sigma X) = [\Sigma^{\infty+1} X, E]_{-n-1} \cong [\Sigma^\infty X, E]_{-n} = \widetilde{E}^n(X).$$

Here the spectrum $E \wedge X$ is obtained by taking the smash product of E with X term by term, and the structure maps are induced by those of E.

Remark 1.2.13. *In fact, one may generalize the definition of $E \wedge X$ and define the smash product $E \wedge F$ of any two spectra E and F. However, one should be aware that this definition is valid only in the stable homotopy category. One can consult either Adams' book [2] or Kochman's book [6] for two concrete constructions of $E \wedge F$. Moreover, there are also many other ways to construct it concretely, and they all turn out to be the same element in the stable homotopy category.*

Remark 1.2.14. *Recall that any cohomology theory leads to an Ω-spectrum. However, in the opposite direction, the construction of a cohomology theory from a spectrum that we give above does not require the spectrum to be an Ω-spectrum. This is not surprising since one can prove that any spectrum is homotopy equivalent to an Ω-spectrum, and two homotopy equivalent spectra give rise to the same cohomology theory.*

In particular, we can take E to be the Thom spectrum MU that we have constructed before. Its associated homology and cohomology theories will be denoted by MU_* and MU^* respectively.

1.3 Formal group laws

Similar to the ordinary cohomology, we can prove that the ring $MU^*(BU)$ is a formal power series ring with coefficient $\pi_*(MU)$ and one variable in each even degree. Here, $BU := \varinjlim BU(n)$ is defined to be the colimit of the classifying spaces. If we use cf_k to denote the variable in the $2k$-th degree for $k \in \mathbb{Z}_{>0}$, then we have

$$MU^*(BU) = \pi_*(MU)[[cf_1, cf_2, ..., cf_n, ...]].$$

The classes cf_k, $k \in \mathbb{Z}_{k>0}$, are called the Conner–Floyd classes, and are the counterparts of the Chern classes in the ordinary cohomology. So in particular, we can define the Euler class of a complex line bundle L in cohomology MU^* to be its first Conner–Floyd class, and we still denote it by $e(L)$. For every $n \in \mathbb{N}$, we have

$$MU^*(BU(n)) = \pi_*(MU)[[i_n^*(cf_1), ..., i_n^*(cf_n)]],$$

where $i_n : BU(n) \hookrightarrow BU$ is the inclusion. Particularly, in the case when $n = 1$, we have $BU(1) \cong \mathbb{CP}^\infty$ and

$$MU^*(\mathbb{CP}^\infty) \cong \pi_*(MU)[[x]],$$

where $x = e(\mathcal{O}(-1)) \in MU^2(\mathbb{CP}^\infty)$ is the Euler class of the tautological bundle over $\mathbb{CP}^\infty$. We can also compute

$$MU^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \pi_*(MU)[[x_1, x_2]],$$

where $x_i = P_i^*(x)$ and $P_i : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$, $i = 1, 2$, are the two projection maps.

In the following, we will construct an extra algebraic structure on $\pi_*(MU)$, called a formal group law. To do this, we consider the Euler class of the complex line bundle defined by $P_1^*\mathcal{O}(-1) \otimes P_2^*\mathcal{O}(-1)$ over $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$. Then it is a formal power series $F(x_1, x_2)$ in x_1 and x_2 with coefficients in $\pi_*(MU)$ by our computation of $MU^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$. Since $\mathcal{O}(-1)$ is the universal line bundle and any other line bundle can be pulled back from it, one shows easily that we have

$$e(L_1 \otimes L_2) = F(e(L_1), e(L_2)) \tag{1}$$

for any two line bundles L_1 and L_2 with the same base manifold.

Since the tensor products of line bundles are associative and commutative, we have

$$F(F(x,y),z) = F(x,F(y,z)) \text{ and } F(x,y) = F(y,x).$$

By taking one of the line bundles to be the trivial line bundle in the equation 1, we get

$$F(x,0) = F(0,x) = x.$$

This leads to the definition of a formal group law as follows.

Definition 1.3.1. *Let R be a commutative ring. Then a formal group law over R is a formal power series $F \in R[[x,y]]$ satisfying the following requirements*

- *$F(x,0)=F(0,x)=x$,*
- *$F(F(x,y),z)=F(x,F(y,z))$.*

Moreover, if F satisfies $F(x,y) = F(y,x)$, then we say that F is a commutative formal group law over R.

Therefore, we have defined a commutative formal group law F over the coefficient ring $\pi_*(MU)$. We denote by C the subring of $\pi_*(MU)$ generated by the coefficients of F.

In 1955, Lazard proved in his article [8] that there exists a universal commutative formal group law F_{univ} over a universal ring $\mathbb{L}$, called Lazard's ring. That is to say, they satisfy the following universal property: for any commutative ring R, there is a bijection between commutative formal group laws over R and ring homomorphisms from $\mathbb{L}$ to R. And the bijection is given by mapping the ring homomorphism $f : \mathbb{L} \to R$ to the formal group law $f \circ F_{\text{univ}}$ over R. Another way of expressing this fact is as follows. Let FGL be the functor that sends any commutative ring to the set of formal group laws over it. Then it is corepresented by $\mathbb{L}$.

Remark 1.3.2. *In general, we can replace MU^* by any complex-oriented cohomology theory E^*. By definition, the complex orientation of E^* defines the first Conner–Floyd class $c_1^E \in E^2(BU(1))$. The existence of such a class guarantees that the Atiyah–Hirzebruch spectral sequence of $BU(1)$ collapses, and we have*

$$E^*(BU(1)) = \pi_*(E)[[c_1^E]].$$

Furthermore, this leads to the definition of all higher Conner–Folyd classes. In particular, this means that we also have a formal group law over $\pi_*(E)$ for any complex-oriented cohomology theory E^*.

As is shown by Ravenel in Lemma 4.1.13 of his book [12], MU^* is the universal complex-oriented cohomology theory. Therefore, we may expect the formal group law F over $\pi_*(MU)$ to be the universal formal group law, and the ring $\pi_*(MU)$ to be the Lazard ring. This is indeed true, and is named as the Milnor–Quillen Theorem. We will give a proof of it in Section 4.3.

2 Geometric interpretations and basic properties

In 1953, Thom showed in his paper [13] that both oriented and unoriented cobordism rings can be computed as the homotopy groups of the corresponding Thom spectra. Moreover, Thom's proof can be generalized naturally to any $(\mathfrak{B}, f)$-structure, which endows vector bundles with an extra structure, and its associated Thom spectrum. This construction is now called the Thom–Pontryagin construction. In this article, we are particularly interested in the case where the $(\mathfrak{B}, f)$-structure is taken to be the stable complex structure on vector bundles, and its associated Thom spectrum turns out to be MU that we have defined before in Section 1.2. Therefore, by using Thom–Pontryagin construction, we are able to give a geometric interpretation to the homology group $MU_*(X)$ for any topological space X. This will be explained in detail in the first subsection.

Since we can also define a cohomology theory MU^* from the spectrum MU, a natural question arises: does $MU^*(X)$ have a geometric interpretation in a similar way? We will answer this question in the second subsection. To this end, we will need to restrict our X to be a closed smooth submanifold of some Euclidean space, possibly with a boundary. But this does not lose much generality, since any finite CW-complex has the homotopy type of such a manifold. We restrict all manifolds that appear in this article to be of this kind unless otherwise stated. Before diving into $MU^*(X)$, let's carry out the Thom–Pontryagin construction and recall the geometric meaning of $MU_*(X)$.

2.1 Geometric interpretation of $MU_*(X)$

Definition 2.1.1. *A weakly complex manifold is a manifold M with a complex structure on the vector bundle $TM \oplus \mathbb{R}^n$ for some $n \in \mathbb{N}$. Here, $\mathbb{R}^n$ is understood as the trivial real vector bundle of rank n over M.*

Of course, we need to do some identifications. We identify the complex structure on $TM \oplus \mathbb{R}^n$ with the naturally induced complex structure on the bundle $TM \oplus \mathbb{R}^{n+2} \cong TM \oplus \mathbb{R}^n \oplus \mathbb{R}^2$, where $\mathbb{R}^2$ carries the canonical complex structure via the natural isomorphism $\mathbb{R}^2 \cong \mathbb{C}$.

Let M and M' be two closed and weakly complex manifolds of the same dimension n. And let X be a topological space. We say that two continuous maps $f : M \to X$ and $f' : M' \to X$ are bordant if and only if there exists a weakly complex manifold W of dimension $n + 1$ with boundary $\partial W = M \sqcup M'$ and a continuous map $g : W \to X$ such that the restrictions of g to M and M' are f and f', and that the induced weakly complex structures on M and M' from W coincide with the original ones. This defines an equivalence relation.

Definition 2.1.2. *Let X be a topological space and n be any non-negative integer, then we define the complex bordism group $\Omega_n^U(X)$ to be the bordism classes of maps $M \to X$ where M is a closed weakly complex manifold of dimension n.*

The group structure of $\Omega_*^U(X)$ is defined as follows. For any two continuous maps $f : M \to X$ and $f' : M' \to X$ in $\Omega_*^U(X)$, we define their sum to be the bordism class of $f \sqcup f' : M \sqcup M' \to X$ where $M \sqcup M'$ is equipped with the induced weakly complex structure. Moreover, when $X = pt$, we can also equip the group $\Omega_*^U := \Omega_*^U(pt)$ with a product structure, making it into a ring. In this case, Ω_*^U is just the bordism classes of closed weakly complex manifolds since there is always a unique map from such a manifold to a point. The product of two elements represented by M and M' is given by the bordism class of their product space $M \times M'$ with the induced weakly complex structure. And the unit element is the bordism class of a point with a trivial weakly complex structure on it.

Now we recall the Thom–Pontryagin construction which identifies $MU_*(X)$ with the bordism group $\Omega_*^U(X)$. We start by defining a map

$$\delta : \Omega_*^U(X) \to MU_*(X).$$

Let M be a closed weakly complex manifold of dimension n equipped with a continuous map $f : M \to X$. Then $TM \oplus \mathbb{R}^r$ admits a complex structure for some $r \in \mathbb{N}$. We can embed it into a trivial complex vector bundle $M \times \mathbb{C}^N$ for some N. Since M is closed and of finite dimension, we can take N big enough and assume that M can be embedded it into $A \cong \mathbb{R}^{2N} \oplus \mathbb{R}^r$. We denote the normal bundle of M in A to be ν. Then we have

$$\nu \cong (\mathbb{R}^{2N} \oplus \mathbb{R}^r)/TM \cong (\mathbb{R}^{2N} \oplus \mathbb{R}^r \oplus \mathbb{R}^r)/(TM \oplus \mathbb{R}^r) \cong \mathbb{C}^r \oplus (\mathbb{C}^N/(TM \oplus \mathbb{R}^r)),$$

where we identify $\mathbb{R}^r \oplus \mathbb{R}^r$ with $\mathbb{C}^r$ canonically and identify $\mathbb{R}^{2N}$ with the trivial bundle $\mathbb{C}^N$ that containes $TM \oplus \mathbb{R}^r$. Since $TM \oplus \mathbb{R}^r$ is a complex subbundle of $\mathbb{C}^N$, there is an induced complex structure on $\mathbb{C}^N/(TM \oplus \mathbb{R}^r)$, and hence on ν. Denote by $k = N + (r - n)/2$, then ν is a complex vector bundle of rank k. Since $TM \oplus \mathbb{R}^r$ admits a complex structure, r and n must have the same parity, and hence k is an integer. Now, the complex vector bundle ν is classified by a map $\gamma : M \to BU(k)$ where $EU(k) \to BU(k)$ is the universal complex vector bundle of rank k. And we get the following diagram

$$\begin{array}{ccc} \nu & \xrightarrow{\gamma'} & EU(k) \\ \downarrow & & \downarrow \\ M & \xrightarrow[\gamma]{} & BU(k). \end{array}$$

Choose an open tubular neighborhood N of M in A. And denote by A^* the one-point compactification of A. Since M is closed, the closure of N is compact in A, and we have a natural map

$$A^* \to A/(A - N) \cong \bar{N}/\partial N \cong T(\nu),$$

where $T(\nu)$ is the Thom space of ν. Now, we use $D(\nu)$ and $S(\nu)$ to denote the disc bundle and sphere bundle associated with ν. Then we can construct a map $D(\nu) \to X \times D(EU(k))$, whose first component is the composition of the projection $D(\nu) \to M$ with $f : M \to X$ and the second component is induced from γ'. Similarly, we have a map $S(\nu) \to X \times S(EU(k))$. Combining these two maps, we get a map

$$T(\nu) = \frac{D(\nu)}{S(\nu)} \to \frac{X \times D(EU(k))}{X \times S(EU(k))} \cong X_+ \wedge \frac{D(EU(k))}{S(EU(k))} = X_+ \wedge MU(k).$$

Composing this map with $A^* \to T(\nu)$ that we have constructed before, we get a map $A^* \to X_+ \wedge MU(k)$. Since $A^* \cong \mathbb{S}^{2N+r}$, this map represents an element in $\pi_{2N+r}(X_+ \wedge MU(k)) = \pi_{2N+r}(X_+ \wedge MU_{2k})$. By passing to the colimit, we get an element in $\pi_n(X_+ \wedge MU) = MU_n(X)$ since $2N + r - 2k = n$. Therefore, we have defined a map $\delta : \Omega^U_*(X) \to MU_*(X)$.

Next, we prove that this map is well-defined. Suppose that M and M' are two closed weakly complex manifolds of dimension n that are bordant. Suppose that the bordism is given by $g : W \to X$, where W is a weakly complex manifold of dimension $n+1$. Since ∂W is equal to the disjoint union of M and M', there exists an embedding $e : W \hookrightarrow A \times \mathbb{R}$ where $A \cong \mathbb{C}^{n+k}$ is some Euclidean space such that e is transversal to $A \times \{i\}$, $i = 0, 1$, and the pullbacks of them along the embedding e coincide with M and M' respectively. We denote the normal bundle of W in $A \times \mathbb{R}$ to be ω, and the normal bundles of M, M' in $A \times \{0\}, A \times \{1\}$ to be ν, ν'. Since the weakly complex structure of w restricted to the boundary coincides with ν and ν', the classifying maps fit into the following diagram

$$\begin{array}{ccccc} M & \xhookrightarrow{i} & W & \xhookleftarrow{i'} & M' \\ & \searrow & \downarrow & \swarrow & \\ & & BU(k). & & \end{array}$$

Recalling the Thom–Pontryagin construction for ν, ν' and ω, we see that the diagram above leads to the following diagram

$$\begin{array}{ccccc} A^* \times \{0\} & \longrightarrow & T(\nu) & & \\ \downarrow & & \downarrow & \searrow & \\ A^* \times \mathbb{R} & \longrightarrow & T(w) & \longrightarrow & X_+ \wedge MU(k), \\ \uparrow & & \uparrow & \nearrow & \\ A^* \times \{1\} & \longrightarrow & T(\nu') & & \end{array}$$

which gives a homotopy between the maps associated with M and M'. Therefore, they give rise to the same element in $MU_{2n}(X)$ and the map δ is well-defined.

The next step is to construct the inverse map of δ.

Let α be an element in $MU_{2n}(X)$. Since we have

$$MU_{2n}(X) = \pi_{2n}(X_+ \wedge MU) = \varinjlim \pi_{2n+2k}(X_+ \wedge MU(k)),$$

the element α can be represented by some map $g : \mathbb{S}^{2n+2k} \to X_+ \wedge MU(k)$ for some k. Recall that $MU(k)$ is the Thom space of $EU(k)$, we have a natural embedding of the zero section into it, namely $j : BU(k) \hookrightarrow MU(k)$. And it induces an embedding $j' : X \times BU(k) \hookrightarrow X_+ \wedge MU(k)$. By Thom's transversality theorem, we may assume that j' is transversal to g. Then we take the pullback of j' along g and get the following diagram

$$\begin{array}{ccc} M & \xrightarrow{g'} & X \times BU(k) \\ {\scriptstyle i'}\big\downarrow & & \big\downarrow{\scriptstyle j'} \\ \mathbb{S}^{2n+2k} & \xrightarrow[g]{} & X_+ \wedge MU(k). \end{array}$$

We denote by $f : M \to X$ the composition of g' with the projection onto X. Now, to get an element in $\Omega_*^U(X)$, we only need to recover the weakly complex structure on M. Since the basepoint of $X_+ \wedge MU(k)$ is not contained in the image of j', the basepoint of $\mathbb{S}^{2n+2k}$ is not contained in the image of i'. Therefore, it factors through $i : M \hookrightarrow \mathbb{R}^{2n+2k}$. Now, we only need to give a complex structure on the normal bundle $\nu_i = \nu_{i'}$. Since we have assumed that j' is transversal to g, the normal bundle $\nu_{i'}$ is isomorphic to the pullback of the normal bundle $\nu_{j'}$ along g'. Since j' is induced by the inclusion of the zero section into the Thom space $EU(k)$, we have $\nu_j' \cong X \times EU(k)$. Therefore, the left square of the following diagram is a pullback square.

$$\begin{array}{ccccc} \nu_i = \nu_{i'} & \longrightarrow & X \times EU(k) & \longrightarrow & EU(k) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ M & \longrightarrow & X \times BU(k) & \longrightarrow & BU(k) \end{array}$$

The two horizontal maps in the right square are projections onto the second components. Since the right square of this diagram is also a pullback square, the outer square is still a pullback square, making ν_i into a complex vector bundle.

Therefore, we have defined a map $MU_*(X) \to \Omega_*^U(X)$. One can check that this map is well-defined and is indeed the inverse of δ.

2.2 Geometric interpretation of $MU^*(X)$

After recalling the Thom–Pontryagin construction above, we are ready to give a similar explanation to $MU^*(X)$ by using the so-called "complex orientation" instead of a weakly complex structure. Now, we have to restrict our X to be a manifold throughout this subsection. For a map $f : Z \to X$ between manifolds, we define the dimension of f at $z \in Z$ to be the dimension of Z at z minus the dimension of X at $f(z)$.

Definition 2.2.1. *Let $f : Z \to X$ be a map between manifolds such that the dimension of f is everywhere even. Then a complex orientation of f is a factorization $f = p \circ i : Z \hookrightarrow E \to X$, where $i : Z \hookrightarrow E$ is an embedding with a* *complex structure on its normal bundle ν_i, and $p : E \to X$ is a complex vector bundle over X.*

Of course, we also need to do some identifications here. Two complex orientations, denoted by E and E', are equivalent if and only if they can be embedded in a common complex vector bundle E'' over X such that i and i' are isotopic in the common vector bundle E''. That is to say, we have the following diagram

$$\begin{array}{ccccc}
Z \times \{0\} & \xhookrightarrow{i} & E \times \{0\} & \overset{p}{\searrow} & \\
\downarrow & & \downarrow & & \\
Z \times I & \xhookrightarrow{i''} & E'' \times I & \longrightarrow & X. \\
\uparrow & & \uparrow & & \\
Z \times \{1\} & \xhookrightarrow[i']{} & E' \times \{1\} & \underset{p'}{\nearrow} &
\end{array}$$

We also require $\nu_{i''}$ to carry a complex structure such that its restrictions to the ends can be identified with the complex structures on ν_i and $\nu_{i'}$.

In the case when the dimension of f is everywhere odd, we simply replace E by $E \times \mathbb{R}$. That is to say, we still require E to be a complex vector bundle over X, while the normal bundle of the inclusion $Z \hookrightarrow E \times \mathbb{R}$ should admit a complex structure. For a general map $f : Z \to X$, we divide it into the even-dimensional part $f' : Z' \to X$ and the odd-dimensional part $f'' : Z'' \to X$ where $Z = Z' \sqcup Z''$. Then we define a complex orientation of f to be one on f' and one on f''.

Remark 2.2.2. *If we define the "virtual" normal bundle of a map $f : Z \to X$ to be $\nu_f := f^*TX - TZ$ in $KO(Z)$. Then a complex orientation of f can be viewed as a stable complex structure on ν_f. This is because if we have a factorization $f = p \circ i : Z \hookrightarrow E \to X$ as before, then we have $\nu_f = \nu_i - f^*E$. Since ν_i and f^*E are both complex vector bundles over Z, v_f is a well-defined element in $K(Z)$ for any complex-oriented map f.*

Remark 2.2.3. *By embedding into a bigger complex vector bundle, we may assume that our complex vector bundle E over X in the definition above is trivial. By doing so, one can easily define the pullback of a complex orientation and the composition of complex orientations.*

As before, we can define when two proper complex-oriented maps are cobordant. Let $f_i : Z_i \to X$, $i = 0, 1$ be two proper complex-oriented maps. They are cobordant if there exists a proper complex-oriented map $b : W \to X \times \mathbb{R}$ such that it is transversal to $\epsilon_i : X \to X \times \mathbb{R}$, $\epsilon_i(x) = (x, i)$ for $i = 0, 1$. And the pullback of b along ϵ_i coincides with f_i and gives the same complex orientation. This defines an equivalence relation between proper complex-oriented maps.

Definition 2.2.4. *Let X be a manifold, then we define the complex cobordism ring $\Omega^q_U(X)$ to be the set of cobordism classes of proper complex-oriented maps to X of dimension $-q$.*

The ring structure of $\Omega_U^*(X)$ is given as follows. Let $f_i : Z_i \to X$, $i = 1, 2$, be two proper complex-oriented maps that represent two elements of $\Omega_U^*(X)$. Then the sum of them is defined by the cobordism class of $f_1 \sqcup f_2 : Z_1 \sqcup Z_2 \to X$ with the induced complex orientation. The product is more complicated and will be postponed until we define the pullback maps of Ω_U^*.

Similar to the case of the homology MU_*, we will prove $MU^*(X) \cong \Omega_U^*(X)$ for any manifold X, which gives a geometric interpretation to $MU^*(X)$. As in the proof of $MU_*(X) \cong \Omega_*^U(X)$, we first construct a map δ from $\Omega_U^*(X)$ to $MU^*(X)$ as follows. Let $f : Z \to X$ be a complex-oriented map, and suppose that its complex orientation is given by the factorization $f = p \circ i$ where the map $i : Z \hookrightarrow E$ is an embedding with a complex structure on ν_i, and $p : E \to X$ is a trivial complex vector bundle of rank r. Then, as in the Thom–Pontryagin construction, we have the collapse map $E^* \to T(\nu_i)$, where E^* is the one-point compactification of E. Since E is a trivial complex vector bundle of rank r over X, we have $E^* \cong X_+ \wedge \mathbb{S}^{2r}$. Meanwhile, since ν_i is a complex vector bundle of rank $s := \dim(E) - \dim(Z)$, it is classified by a map $Z \to BU(s)$. And from the bundle map $\nu_i \to EU(s)$, we get a map $T(\nu_i) \to T(EU(s)) = MU(s)$. By composing this map with the collapse map, we get a map

$$X_+ \wedge \mathbb{S}^{2r} \cong E^* \to T(\nu_i) \to MU(s),$$

which represents an element in $[X_+ \wedge \mathbb{S}^{2r}, MU(s)]$. By passing to the colimit again, we can take its image in $[\Sigma^\infty X_+, MU]_{2r-2s}$ and get an element in the cohomology ring $\widetilde{MU}^{2s-2r}(X_+) \cong MU^{2s-2r}(X)$. Moreover, one can check that we have $2s - 2r = -dim(f) = q$. Therefore, we have defined a map δ from $\Omega_U^*(X)$ to $MU^*(X)$. The well-definedness of δ can be verified similarly to the case of the map $\Omega_*^U(X) \to MU_*(X)$ and will be omitted here.

The next step is to construct the inverse map of δ. Since we have

$$MU^*(X) = \varinjlim [\Sigma^\infty X_+, MU]_{-*} = \varinjlim [\Sigma^k X_+, MU_{k+*}],$$

every element of $MU^{\mathrm{ev}}(X)$ can be represented by some map

$$g : X_+ \wedge \mathbb{S}^{2r} \to MU(s)$$

for some $r, s \in \mathbb{N}$. Since $MU(s)$ is the Thom space of the universal complex vector bundle $EU(s) \to BU(s)$ of rank s, we have the inclusion of the zero section $j : BU(s) \hookrightarrow MU(s)$. By Thom's transversality theorem, we may assume that j is transversal to g. Then we can pull back j along g and get a map $i' : Z \hookrightarrow X_+ \wedge \mathbb{S}^{2r}$. We define E to be the trivial complex bundle over X of rank r. Then we may identify $X_+ \wedge \mathbb{S}^{2r}$ with E^*, the one-point compactification of E. Since $BU(s)$ does not contain the basepoint of $MU(s)$, Z does not contain the basepoint of E^*. Therefore, the embedding i' factors through E and we use i to denote $i : Z \hookrightarrow E$. The composition of this map with the projection $E \to X$ defines a map $f : Z \to X$. For clarity, all the maps above are concluded in the following diagram

$$\begin{array}{ccccc}
Z & = & Z & \longrightarrow & BU(s) \\
{\scriptstyle i}\downarrow & & {\scriptstyle i'}\downarrow & & {\scriptstyle j}\downarrow \\
E & \hookrightarrow & E^* \cong X_+ \wedge \mathbb{S}^{2r} & \xrightarrow[g]{} & MU(s).
\end{array}$$

The next step is to recover the complex orientation of f. Since E is already a (trivial) complex vector bundle over X, we only need to give a complex structure on $\nu_i = \nu_{i'}$. Since j is transversal to g, the normal bundle $\nu_{i'}$ is isomorphic to the pullback of the normal bundle ν_j. Recall that $j : BU(s) \hookrightarrow MU(s)$ is the inclusion of the zero section into the Thom space, the normal bundle ν_j is nothing other than the universal complex vector bundle $EU(s) \to BU(s)$. Therefore, $\nu_i = \nu_{i'}$, being its pullback along $Z \to BU(s)$, is equipped with a complex structure. Thus, we have defined a map $MU^{\text{ev}}(X) \to \Omega_U^{\text{ev}}(X)$. The case when the dimension is odd can be constructed similarly. And the proof that it is inverse to δ is straightforward.

Remark 2.2.5. *Since a complex orientation of a map $f : M \to *$ is the same as a weakly complex structure on M, we have $\Omega_*^U \cong \Omega_U^{-*}$ which is a graded ring isomorphism. This coincides with the fact that*

$$MU_*(pt) \cong \pi_*(MU) \cong MU^{-*}(pt)$$

which is valid for any generalized (co)homology theories constructed from spectra.

The pullbacks and Gysin homomorphisms in $MU^*(X)$ can also be interpreted in the complex cobordism setting. Let $g : Y \to X$ be a map between manifolds and let $[f]$ be an element in $\Omega_U^*(X)$ that is represented by a proper complex-oriented map $f : Z \to X$. Then by Thom's transversality theorem, we may assume that g is transversal to f. Now we pull back f along g and get a map f'. Then the complex orientation of f can also be pulled back and induces a complex orientation on f'. Moreover, since f is proper, f' is also proper. And it turns out that the cobordism class of f' is the pullback of $[f]$ in $\Omega_U^*(Y)$. Now, we are ready to define the product of two elements in $\Omega_U^*(X)$. Suppose that they are represented by two proper complex-oriented maps $f_i : Z_i \to X$, $i = 1, 2$. First, we take the product of them and get the map $f_1 \times f_2 : Z_1 \times Z_2 \to X \times X$ with the induced complex orientation. By taking the cobordism class of it, we get an element $[f_1 \times f_2]$ in $\Omega_U^*(X \times X)$. To get an element in $\Omega_U^*(X)$, we simply pull back this element along the diagonal embedding $\Delta : X \hookrightarrow X \times X$. And we define the product $[f_1][f_2]$ to be the resulting element in $\Omega_U^*(X)$. One can check without difficulty that the unit is given by the cobordism class of the identity map $[id_X] \in \Omega_U^0(X)$.

As for the Gysin homomorphisms, let $g : Y \to X$ be a proper complex-oriented map of dimension d. Then it induces a map $g_* : \Omega_U^*(Y) \to \Omega_U^{*-d}(X)$ given by sending $[f : Z \to Y] \in \Omega_U^*(Y)$ to $[g \circ f : Z \to Y \to X] \in \Omega_U^{*-d}(X)$, where the complex orientation of $g \circ f$ is the composition of the complex orientations of g and f.

By geometric interpretations, the following useful lemmas can be proved directly.

Lemma 2.2.6. *Let*

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{g'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & X \end{array}$$

be a Cartesian square, then we have

$$g^* f_* = f'_* g'^* : MU^*(Y) \to MU^{*-d}(Z),$$

where $d = \dim(f) = \dim(f')$.

Lemma 2.2.7. *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two complex-oriented maps. Then we have*

$$(g \circ f)_* = g_* \circ f_* : MU^*(X) \to MU^{*-\dim(f)-\dim(g)}(Z),$$

where the map $g \circ f$ *is endowed with the composite complex orientation.*

3 Cohomological operations

In this section, we will define two cohomological operations that will appear in the proof of Quillen's Theorem and are of interest of their own. The first one is the Steenrod k-th power operation. We will define it using permutation group actions. The second one is the Landweber–Novikov operation. It is introduced as a direct consequence of a characteristic class in MU^* that mimics the Chern polynomial in the ordinary cohomology theory.

3.1 The Landweber–Novikov operation

As is explained in Section 1.3, for any complex line bundle $L \to X$, we can define its Euler class $e(L) \in MU^2(X)$ in the complex cobordism ring of X. And the definition is also given by $e(L) := i^* i_* 1 \in MU^2(X)$, where $i : X \hookrightarrow L$ is the inclusion of the zero section and $1 \in MU^0(X)$ is the unit element. Now, we can mimic the definition of the Chern polynomial in the ordinary cohomology and give the following definition.

Definition 3.1.1. *For any complex vector bundle* $E \to X$*, we have an associated characteristic class* $c_t(E) \in MU^*(X)[[t]]$*, which is a formal power series in a sequence of variables* $t = (t_1, t_2, ...)$ *and satisfies the following properties:*

- c_t *is natural, i.e., we have* $c_t(f^*E) = f^* c_t(E)$,

- *c_t is additive, i.e., we have $c_t(E_1 \oplus E_2) = c_t(E_1) \cdot c_t(E_2)$,*
- *its value on any line bundle L is given by $c_t(L) = 1 + \sum_{i \geqslant 1} e(L)^i \cdot t_i$.*

As is proved in Section 3.3 of Tamaki and Kono's book [7], the splitting principle still holds in the cohomology theory MU^*, and therefore one can verify that the characteristic class c_t exists and is unique.

Now, we are ready to define the Landweber–Novikov operation. Recall that we use $v_f = f^*TX - TZ \in K(Z)$ to denote the "virtual" normal bundle for any complex-oriented map $f : Z \to X$ in Remark 2.2.2. Clearly, the definition of the characteristic class c_t can be naturally extended to the K-group and therefore defines a map

$$c_t : K(X) \to MU^*(X)[[t]].$$

In particular, $c_t(v_f)$ is meaningful even when f is not necessarily an embedding.

Definition 3.1.2. *The Landweber–Novikov operation*

$$s_t : MU^*(X) \to MU^*(X)[[t]]$$

is defined by

$$s_t(f_*1) = f_*(c_t(\nu_f)),$$

where $f : Z \to X$ is any proper complex-oriented map.

We can write $c_t(E)$ as

$$c_t(E) = \sum_\alpha c_\alpha(E) \cdot t^\alpha$$

where $\alpha := (\alpha_1, \alpha_2, ...)$ and $t^\alpha := t_1^{\alpha_1} t_2^{\alpha_2} \cdots$ for $\alpha_i \in \mathbb{N}$. Similarly, we can expand the operation s_t as

$$s_t(x) = \sum_\alpha s_\alpha(x) \cdot t^\alpha.$$

Then each coefficient s_α is a map $s_\alpha : MU^*(X) \to MU^*(X)$, and is given by $s_\alpha(f_*1) = f_*(c_\alpha(v_f))$ from the definition above.

Moreover, the Landweber–Novikov operator $s_t : MU^*(X) \to MU^*(X)[[t]]$ satisfies the following Riemann–Roch type formula.

Proposition 3.1.3. *Let $f : Z \to X$ be a proper complex-oriented map. Then we have*

$$s_t(f_*z) = f_*(c_t(\nu_f) \cdot s_t z) \in MU^*(X)[[t]]$$

for any $z \in MU^*(Z)$.

Proof. Using the geometric interpretation, we may suppose that z is represented by a proper complex-oriented map $g : W \to Z$. Then we have $z = g_*1$ where $1 \in MU^0(W)$ is the unit and $g_* : MU^*(W) \to MU^{*-\dim(g)}(Z)$ is the Gysin homomorphism. Then we have

$$s_t(f_*z) = s_t(f_*g_*1) = s_t((f \circ g)_*1) = (f \circ g)_* c_t(\nu_{f \circ g})$$

by the definition of s_t. Since the normal bundle $\nu_{f\circ g}$ can be computed as

$$\nu_{f\circ g} = (f\circ g)^*(TX) - TW = g^*(f^*TX - TZ) + (g^*TZ - TW) = g^*\nu_f + \nu_g,$$

we have

$$\begin{aligned}(f\circ g)_*(\nu_{f\circ g}) &= f_*g_*c_t(g^*\nu_f + \nu_g)\\ &= f_*g_*(g^*c_t(\nu_f)\cdot c_t(\nu_g))\\ &= f_*(c_t(\nu_f)\cdot g_*c_t(\nu_g))\\ &= f_*(c_t(\nu_f)\cdot s_t z).\end{aligned}$$

Here, the last equation comes from

$$s_t z = s_t(g_*1) = g_*c_t(\nu_g)$$

by the definition of s_t. □

3.2 The Steenrod operation

Now we define the Steenrod k-th power operation. To do this, we first fix a principal $\mathbb{Z}/k$-bundle $Q \to B$, and let h^* be a $\mathbb{Z}/k$-equivariant cohomology theory defined by $h^*(X) := MU^*(X \times_{\mathbb{Z}/k} Q)$ for any $\mathbb{Z}/k$-manifold X.

For any manifold X, although X itself is not necessarily a $\mathbb{Z}/k$-manifold, there is always a natural action of $\mathbb{Z}/k$ on X^k by permuting the coordinates. We may define the exterior Steenrod operation $P_{\text{ext}} : MU^{-2q}(X) \to h^{-2qk}(X)$ by

$$P_{\text{ext}}(f_*1) = f_*^k(1),$$

where $f : Z \to X$ is any proper complex-oriented map and $f^k : Z^k \to X^k$ is the induced map. Here, f_*^k is the Gysin homomorphism in h^*, which is induced from the Gysin homomophisms in MU^*. First, we show that P_{ext} is well-defined.

Suppose that $f_0 : Z_0 \to X$ and $f_1 : Z_1 \to X$ are two complex-oriented maps that represent the same element in $MU^{-2q}(X)$. Then there exists a complex-oriented map $b : W \to X\times\mathbb{R}$ such that it is transversal to $\epsilon_i : X \to X\times\mathbb{R}$ defined by $\epsilon_i(x) = (x, i)$ for $i = 0, 1$. And the pullback of b along ϵ_i coincides with f_i and gives the same complex orientation. Therefore, we have the following diagram which consists of two Cartesian squares

$$\begin{array}{ccccc} Z_0 & \xrightarrow{\delta_0} & W & \xleftarrow{\delta_1} & Z_1 \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle f_1} \\ X & \xrightarrow[\epsilon_0]{} & X\times\mathbb{R} & \xleftarrow[\epsilon_1]{} & X. \end{array}$$

Since the corresponding diagram with every manifold replaced by its k-th power and every map replaced by the induced one still consists of two Cartesian squares, we have

$$\begin{aligned} P_{\text{ext}}((f_0)_*1) = (f_0^k)_*1 &= (f_0^k)_*(\delta_0^k)^*1 \\ &= (\epsilon_0^k)^* b_*^k 1 \\ &= (\epsilon_1^k)^* b_*^k 1 \\ &= (f_1^k)_*(\delta_1^k)^*1 = (f_1^k)_*1 = P_{\text{ext}}((f_1)_*1), \end{aligned}$$

where we use Lemma 2.2.6 for each Cartesian square and the fact that ϵ_0^k is homotopic to ϵ_1^k. So P_{ext} is well-defined.

Next, we define the (interior) Steenrod operation by the pullback of $P_{\text{ext}}(f_*1)$ along the diagonal embedding. Let $\Delta : X \hookrightarrow X^k$ be the diagonal embedding. We make X into a $\mathbb{Z}/k$-manifold by endowing it with the trivial action. Then Δ becomes a $\mathbb{Z}/k$-map since $\Delta(X) \subseteq X^k$ is invariant under the permutation of coordinates, and we may define

$$P(f_*1) := \Delta^* P_{\text{ext}}(f_*1) \in h^{-2qk}(X).$$

Since $\mathbb{Z}/k$ acts on X trivially, we have

$$h^*(X) = MU^*(X \times_{\mathbb{Z}/k} Q) = MU^*(X \times B).$$

Therefore, the k-th Steenrod operation P maps $MU^{-2q}(X)$ to $MU^{-2qk}(X \times B)$.

3.3 A Grothendieck–Riemann–Roch type formula

Definition 3.3.1. *Let Y and Z be two submanifolds of X. Denote $W = Y \cap Z$. Then we say that Y and Z intersect cleanly along W if $TW = TY|_W \cap TZ|_W$. In this case, we define $F := TX|_W/(TY|_W + TZ|_W)$ to be the excess bundle. In particular, it is a vector bundle over W and vanishes exactly when Y and Z intersect transversely.*

Proposition 3.3.2. *Let Y and Z be two submanifolds of X which intersect cleanly along W. Denote the embeddings as in the diagram below*

$$\begin{array}{ccc} W & \overset{j'}{\hookrightarrow} & Y \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ Z & \underset{j}{\hookrightarrow} & X. \end{array}$$

Recall that we have Thom isomorphisms

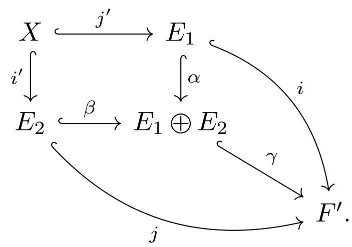
$$\begin{aligned} i_* &: MU^*(Y) \xrightarrow{\sim} MU^{*+a}(X, X - Y), \\ i'_* &: MU^*(W) \xrightarrow{\sim} MU^{*+b}(Z, Z - W), \end{aligned}$$

where a and b are the real ranks of the normal bundles ν_i and $\nu_{i'}$ respectively. They fit into the following equation

$$j^* i_*(y) = i'_*(j'^*(y) \cdot e(F)) \in MU^{*+a}(Z, Z - W)$$

for any $y \in MU^*(Y)$.

Proof. By Lemma 2.2.6, this is a local property and hence we may focus only on the tubular neighborhoods of W in Y and Z. Therefore, we may replace Y and Z with the normal bundles E_1 and E_2. Denote the normal bundle of W in X by F'. Then, by the definition of the excess bundle, we have $F \cong F'/(E_1 \oplus E_2)$. The embeddings fit into the following diagram



Then we have

$$j^* i_*(y) = (\gamma \circ \beta)^* (\gamma \circ \alpha)_*(y) = \beta^* \gamma^* \gamma_* \alpha_*(y) = \beta^*(\alpha_*(y) \cdot e(p_{12}^* F)),$$

where p_{12} is the projection from $E_1 \oplus E_2$ to X. This is because γ is an embedding of vector bundles over X, and therefore we have

$$\nu_\gamma \cong p_{12}^*(F'/(E_1 \oplus E_2)) \cong p_{12}^* F.$$

Further on, we get

$$\beta^*(\alpha_*(y) \cdot e(p_{12}^* F)) = \beta^* \alpha_*(y) \cdot \beta^* p_{12}^* e(F) = i'_* j'^*(y) \cdot p_2^* e(F)$$

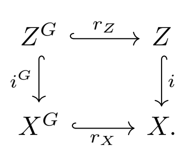
by Lemma 2.2.6 and because $p_{12} \circ \beta = p_2$ is the projection from E_2 to X. Finally, we have

$$i'_* j'^*(y) \cdot p_2^* e(F) = i'_*(j'^*(y) \cdot i'^* p_2^* e(F)) = i'_*(j'^*(y) \cdot e(F))$$

since $p_2 \circ i$ is the identity map id_X. □

Now we apply this proposition to manifolds equipped with group actions and their fixed submanifolds.

Let G be a compact Lie group and X be a G-manifold. Let Z be a G-submanifold of X. Denote by X^G and Z^G the fixed submanifolds of X and Z respectively. And denote the embeddings by the following diagram



Since we have

$$TZ^G = (TZ)^G|_{Z^G} = TZ|_{Z^G} \cap (TX)^G|_{Z^G} = TZ|_{Z^G} \cap TX^G|_{Z^G},$$

we know that Z and X^G intersect cleanly along Z^G. Here, we use the result that for any G-manifold Y, we have $T(Y^G) = (TY)^G|_{Y^G}$. This result can be deduced from Section I.2 of Audin's book [3]. Meanwhile, the excess bundle can be computed as follows

$$\begin{aligned} F = TX/(TZ + TX^G) &\cong (TX/TZ)/(TZ + TX^G/TZ) \\ &\cong (TX/TZ)/(TX^G/TZ \cap TX^G) \\ &\cong (TX/TZ)/(TX^G/TZ^G) \\ &\cong \nu_i/\nu_{i^G}, \end{aligned}$$

where every bundle is understood as a bundle restricted to Z^G. We denote by μ_i the excess buneld ν_i/ν_{i^G}, and we can view it as the non-trivial part of the action of G on ν_i. Fix a principal G-bundle $Q \to B$, we define $h^*(X) := MU^*(X \times_G Q)$ for any G-manifold X. Then h^* is a multiplicative G-equivariant cohomology theory for G-spaces. After taking the product with Q over G, the proof of Proposition 3.3.2 still works and leads to the following corollary.

Corollary 3.3.3. *Let G, X, Z as before. Then we have*

$$r_X^* i_*(z) = i_*^G(r_Z^*(z) \cdot e(\mu_i)) \in h^*(X^G, X^G - Z^G)$$

for any $z \in h^*(Z)$, where $e(\mu_i)$ is the Euler class of μ_i in the cohomology h^*.

It turns out that we can generalize this corollary to any proper complex-oriented G-map $f : Z \to X$, which is not necessarily an embedding, and get the following Grothendieck–Riemann–Roch type formula.

Corollary 3.3.4. *Let G be a compact Lie group and X and Z be two G-manifolds. Let $f : Z \to X$ be a proper complex-oriented G-map. Suppose that the complex orientation of f is given by the factorization $f = p \circ i : Z \hookrightarrow E \to X$, where E is a complex G-bundle over X and i is an embedding with an equivariant complex structure on its normal bundle. Let r_X and r_Z be the embeddings of the fixed submanifolds of X and Z, and let $f^G : Z^G \to X^G$ be the induced map. Then we have*

$$r_X^* f_*(z) \cdot e(\mu_E) = f_*^G(r_Z^*(z) \cdot e(\mu_i)) \in h^*(X^G),$$

where μ_E is the non-trivial part of the G-action on $r_X^* E$ and μ_i is as defined before.

Proof. Let $T(E)$ be the Thom space of the vector bundle $p : E \to X$. Then we have a natural embedding $j : X \hookrightarrow T(E)$ of the zero section. Now, we can apply Corollary 3.3.3 to embeddings i and j with $z \in h^*(Z)$ and $f_*(z) \in h^*(X)$ respectively and get

$$r_E^* i_* z = i_*^G(r_Z^* z \cdot e(\mu_i)), \tag{2}$$
$$r_T^* j_* f_* z = j_*^G(r_X^* f_* z \cdot e(\mu_j)), \tag{3}$$

where $r_T : T(E)^G \hookrightarrow T(E)$ is the embedding of the fixed submanifold. Moreover, μ_j is defined as the non-trivial part of the G-action on $r_X^*(v_j)$. Since v_j is just E, we have $\mu_j = \mu_E$. By the Thom isomorphisms

$$h^*(T(E)) \cong h^*(E, E - X),$$
$$h^*(T(E)^G) \cong h^*(E^G, E^G - X^G),$$

the equation (3) can also be written as

$$r_E^* j_* f_* z = j_*^G (r_X^* f_* z \cdot e(\mu_E)). \tag{4}$$

By the Thom isomorphism, we have

$$j_* f_* = j_*(p \circ i)_* = j_*(p_* \circ i_*) = (j_* \circ p_*) i_* = i_*.$$

Therefore, the left hand sides of equation (2) and equation (4) are the same, and we get

$$i_*^G (r_Z^* z \cdot e(\mu_i)) = j_*^G (r_X^* f_* z \cdot e(\mu_E)).$$

Similarly, we also have $j_*^G f_*^G = i_*^G$, and the equation becomes

$$j_*^G f_*^G (r_Z^* z \cdot e(\mu_i)) = j_*^G (r_X^* f_* z \cdot e(\mu_E)).$$

Therefore, we have

$$f_*^G (r_Z^* z \cdot e(\mu_i)) = r_X^* f_* z \cdot e(\mu_E)$$

since j_*^G is the Thom isomorphism of the vector bundle $p : E \to X$. □

3.4 The relation between the Landweber–Novikov operations and the Steenrod operations

Consider the action of $\mathbb{Z}/k$ on $\mathbb{C}^k$ by permuting the coordinates. We use ρ to denote the induced $\mathbb{Z}/k$-action on the subspace

$$V := \{(z_1, z_2, ..., z_k) | z_1 + z_2 + ... + z_k = 0\} \subseteq \mathbb{C}^k.$$

And we use η to denote the action of $\mathbb{Z}/k$ on $\mathbb{C}$ by multiplication with the k-th roots of unity.

Let $f : Q \to B$ be a principal $\mathbb{Z}/k$-bundle. Then we have the associated Steenrod k-th power operation $P : MU^{-2q}(X) \to MU^{-2qk}(X \times B)$. We have also defined the Landweber–Novikov operation $s_t : MU^*(X) \to MU^*(X)[[t]]$. The proposition below gives the relation between these two operations.

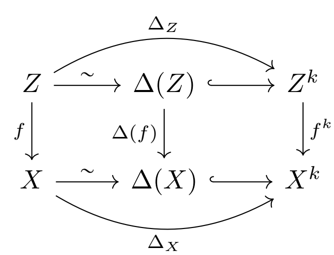
Before stating the proposition, we need a few more definitions. Recall that we have defined the formal group law F over MU^* in Section 1.3. Now, we define $[k]_F(v)$ by applying F to v and itself k times for any $v \in MU^*(X)$. To be more specific, we can define $[k]_F(v)$ inductively on k. We define $[1]_F(v) := v$ and $[k]_F(v) := F([k-1]_F(v), v)$. In particular, we have $e(L^{\otimes k}) = [k]_F(e(L))$ for any complex line bundle L. Finally, recall that $C \subseteq \pi_*(MU) = MU^*(pt)$ is the subring generated by the coefficients of the formal group law F.

Proposition 3.4.1. *Let $f : Q \to B$ be a principal $\mathbb{Z}/k$-bundle. By abuse of notation, we use η and ρ to denote the vector bundles over B associated with the corresponding group actions and f. More concretely, we have $\eta := Q \times_{\mathbb{Z}/k} \mathbb{C} \to B$ being a complex line bundle and $\rho := Q \times_{\mathbb{Z}/k} V \to B$ being a complex vector bundle of rank $k-1$, where the action of $\mathbb{Z}_k$ on $\mathbb{C}$ and V is η and ρ. Denote their Euler classes by $v = e(\eta) \in MU^2(B)$ and $w = e(\rho) \in MU^{2(k-1)}(B)$. Then we have the following relation for any $x \in MU^{-2q}(X)$*

$$w^{n+q}P(x) = \sum_{l(\alpha) \leqslant n} w^{n-l(\alpha)} a(v)^{\alpha} s_{\alpha}(x) \in MU^{2n(k-1)-2q}(X \times B),$$

where n is any integer sufficiently large with respect to the dimension of X and q. Here, for every positive integer j, $a_j(v)$ is some formal power series in v with coefficients in C which will be made concrete in the proof. We use $a(v)^{\alpha}$ to denote $\prod_i a_i(v)^{\alpha_i}$ for $\alpha = (\alpha_1, \alpha_2, ...)$ with all $\alpha_i \in \mathbb{N}$. And for such an α, $l(\alpha)$ is defined to be $l(\alpha) := \sum_i \alpha_i$.

Proof. Assume that $x = f_*1$ is represented by a map $f = p \circ i : Z \hookrightarrow E \to X$. Then by definition, we have $P(x) = \Delta_X^* f_*^k 1$, where $\Delta_X : X \hookrightarrow X^k$ is the diagonal embedding. Now, since $\mathbb{Z}/k$ acts transversely on the coordinates of X^k, the fixed submanifold is $\Delta(X) \subseteq X^k$. Therefore, we can apply Corollary 3.3.4 to the following diagram



and we get

$$\Delta_X^* f_*^k(1) \cdot e(\mu_{E^k}) = f_*(\Delta_Z^*(1) \cdot e(\mu_{i^k})).$$

Since μ_{E^k} is the non-trivial part of the action of $\mathbb{Z}/k$ on $\Delta_X^*(E^k) = E^{\oplus k}$ and this action is in fact the permutation of the direct summands, we have $\mu_{E^k} = E \otimes \rho$. Similarly, we have $\mu_{i^k} = v_i \otimes \rho$. So the equation becomes

$$P(x) \cdot e(E \otimes \rho) = f_* e(v_i \otimes \rho).$$

Moreover, we can take E to be the trivial bundle $m\epsilon$ as long as m is big enough with respect to the dimension of Z. Then in this case, we have $v_i = v_f + m\epsilon$ and $E \otimes \rho = \rho^{\oplus m}$, and we get

$$w^m \cdot P(x) = f_* e((v_f + m\epsilon) \otimes \rho). \tag{5}$$

Now we compute $e(E \otimes \rho)$ for any vector bundle E. First, we do the computation when E is a complex line bundle $E = L$. Since the representation ρ

can be decomposed as $\rho = \bigoplus_{i=1}^{i=k-1} \eta^i$, we have

$$\begin{aligned} e(L \otimes \rho) = e(L \otimes (\bigoplus_{i=1}^{k-1} \eta^i)) &= \prod_{i=1}^{k-1} e(L \otimes \eta^i) = \prod_{i=1}^{k-1} F(e(L), [i]_F(v)) \\ &= \prod_{i=1}^{k-1} ([i]_F(v) + O(e(L))) = w + \sum_{j \geqslant 1} a_j(v) \cdot e(L)^j, \end{aligned}$$

where $a_j(v)$ is some formal power series in v with coefficients in C. Therefore, if E can be decomposed as the direct sum of line bundles $E = \bigoplus_{i=1}^{n} L_i$, we have

$$e(E \otimes \rho) = \prod_{i=1}^{n} e(L_i \otimes \rho) = \prod_{i=1}^{n} (w + \sum_{j \geqslant 1} a_j(v) \cdot e(L_i)^j). \tag{6}$$

Meanwhile, in this case, we recall the definition of the characteristic class c_t and get

$$\sum_{\alpha} c_\alpha(E) \cdot t^\alpha = c_t(E) = \prod_{i=1}^{n} c_t(L_i) = \prod_{i=1}^{n} (1 + \sum_{j \geqslant 1} t_j \cdot e(L_i)^j).$$

Since the right-hand side is the multiplication of n terms, the α that appears on the left-hand side must satisfy $l(\alpha) \leqslant n$. Now we take $t_j = a_j(v)/w$ in the equation above, and get

$$\sum_{l(\alpha) \leqslant n} c_\alpha(E) \cdot \frac{a(v)^\alpha}{w^{l(\alpha)}} = \prod_{i=1}^{n} (1 + \sum_{j \geqslant 1} \frac{a_j(v)}{w} \cdot e(L_i)^j),$$

and hence

$$\sum_{l(\alpha) \leqslant n} w^{n-l(\alpha)} a(v)^\alpha \cdot c_\alpha(E) = \prod_{i=1}^{n} (w + \sum_{j \geqslant 1} a_j(v) \cdot e(L_i)^j). \tag{7}$$

By comparing equation (6) with equation (7), we have proved

$$e(E \otimes \rho) = \sum_{l(\alpha) \leqslant n} w^{n-l(\alpha)} a(v)^\alpha \cdot c_\alpha(E)$$

for vector bundles E that can be split as a direct sum of line bundles. Then this formula can be generalized to any vector bundle E by the splitting principle. Now we get back to equation (5) and use $c_\alpha(v_f + m\epsilon) = c_\alpha(v_f)$, then we get

$$\begin{aligned} w^m \cdot P(x) &= \sum_{l(\alpha) \leqslant m-q} w^{m-q-l(\alpha)} a(v)^\alpha f_* c_\alpha(v_f) \\ &= \sum_{l(\alpha) \leqslant m-q} w^{m-q-l(\alpha)} a(v)^\alpha s_\alpha(x). \end{aligned}$$

If we replace $m-q$ by n, then we get the following equation which is valid for any n big enough with respect to $\dim(X)$ and q:

$$w^{n+q}\cdot P(x)=\sum_{l(\alpha)\leqslant n} w^{n-l(\alpha)}a(v)^{\alpha}s_{\alpha}(x).$$

□

Remark 3.4.2. *From the relation $\rho=\bigoplus_{i=1}^{k-1}\eta^i$, we can express w as a formal power series in v with leading term $(k-1)!\cdot v^{k-1}$. In fact, we have*

$$w=e(\rho)=\prod_{i=1}^{k-1}e(\eta^i)=\prod_{i=1}^{k-1}(iv+O(v^2))=(k-1)!\cdot v^{k-1}+O(v^k).$$

This result will be useful in the proof of Quillen's Theorem.

4 Quillen's Theorem

In this section, we will begin with the last ingredient that we need to prove Quillen's Theorem, which is a technical lemma that is stated and proved in the first subsection. Then we present the proof of Quillen's theorem in detail in the second subsection. The proof begins with the relation between two cohomological operations that we have just studied, and is carried out using a decreasing induction and a localization argument. Finally, as a corollary of Quillen's Theorem, we give the proof of the Milnor–Quillen theorem in the last subsection.

4.1 A technical lemma

The last thing we need to prove Quillen's Theorem is a technical lemma, which can be viewed as a refined version of the Gysin sequence associated with the universal principal $\mathbb{Z}/k$-bundle $\mathbb{S}^\infty\to\mathbb{S}^\infty/(\mathbb{Z}/k)$. Here, we view $\mathbb{S}^\infty$ as the unit sphere in $\mathbb{C}^\infty$, and the action of $\mathbb{Z}/k$ on $\mathbb{S}^\infty$ is induced from the action of $\mathbb{Z}/k$ on $\mathbb{C}^\infty$ via coordinate-wise multiplication with the k-th roots of unity. Fix a weakly complex manifold X, we define a complex-oriented cohomology h^* by $h^*(Y):=MU^*(X\times Y)$. For any map $f:Z\to Y$, its pullback in h^* is denoted by $f^!$ and is defined by

$$f^!=(id_X\times f)^*:h^*(Y)=MU^*(X\times Y)\to MU^*(X\times Z)=h^*(Z).$$

If f is complex-oriented of dimension d, then the associated Gysin homomorphism is denoted by $f_!$ and is defined by

$$f_!=(id_X\times f)_*:h^*(Z)=MU^*(X\times Z)\to MU^{*-d}(X\times Y)=h^{*-d}(Y).$$

The principal $\mathbb{Z}/k$-bundle $\mathbb{S}^\infty\to\mathbb{S}^\infty/(\mathbb{Z}/k)$ has an associated $\mathbb{S}^1$-bundle induced by the action of $\mathbb{Z}/k$ on $\mathbb{S}^1$ by rotation. We denote it by

$$\pi:\mathbb{S}^\infty\times_{\mathbb{Z}/k}\mathbb{S}^1\to\mathbb{S}^\infty/(\mathbb{Z}/k).$$

The Gysin sequence of this bundle has the form

$$h^{q+1}(\mathbb{S}^\infty/(\mathbb{Z}/k)) \xrightarrow{\pi^!} h^{q+1}(\mathbb{S}^\infty \times_{\mathbb{Z}/k} \mathbb{S}^1) \xrightarrow{\pi_!} h^q(\mathbb{S}^\infty/(\mathbb{Z}/k)) \xrightarrow{\cdot v} h^{q+2}(\mathbb{S}^\infty/(\mathbb{Z}/k)),$$

where $v \in h^2(\mathbb{S}^\infty/(\mathbb{Z}/k))$ is the Euler class of the associated line bundle of π in the cohomology h^*. Here, the associated line bundle is given by $\mathbb{S}^\infty \to \mathbb{S}^\infty \times_{\mathbb{Z}/k} \mathbb{C}$ where $\mathbb{Z}/k$ acts on $\mathbb{C}$ by multiplication with the k-th roots of unity.

Now, we consider the finite-dimensional picture. For each positive integer m, we have a principal $\mathbb{Z}/k$-bundle $\mathbb{S}^{2m-1} \to \mathbb{S}^{2m-1}/(\mathbb{Z}/k)$ induced by the restriction of the coordinate-wise action of $\mathbb{Z}/k$ on $\mathbb{C}^m$ to the unit sphere $\mathbb{S}^{2m-1} \subseteq \mathbb{C}^m$. We have the associated line bundle $\mathbb{S}^{2m-1} \times_{\mathbb{Z}/k} \mathbb{C} \to \mathbb{S}^{2m-1}/(\mathbb{Z}/k)$ and we denote its Euler class by $v_m \in h^2(\mathbb{S}^{2m-1}/(\mathbb{Z}/k))$. Obviously, v_m is the finite version of v. We want an analog of the Gysin sequence above in the case of finite dimensional spheres, which will serve as the technical lemma in the proof of Quillen's Theorem. Recall that we have defined $[k]_F$ before stating Proposition 3.4.1, now we define

$$\phi(v) := \frac{[k]_F(v)}{v} = k + O(v)$$

to be a formal power series in v with coefficients in C and the scalar term k. In particular, for any $v \in h^2(Y)$, we have $\phi(v) \in h^0(Y)$.

Lemma 4.1.1. *Let x be an element of $h^q(\mathbb{S}^{2m+1}/(\mathbb{Z}/k))$ satisfying $x \cdot v_{m+1} = 0$. Then we have $j_m^! x = y \cdot \phi(v_m) \in h^q(\mathbb{S}^{2m-1}/\mathbb{Z}_q)$ for some $y \in h^q(pt)$, where j_m is the natural embedding $j_m : \mathbb{S}^{2m-1}/(\mathbb{Z}/k) \hookrightarrow \mathbb{S}^{2m+1}/(\mathbb{Z}/k)$.*

Proof. We consider two families of sphere bundles parametrized by $m \in \mathbb{Z}_{>0}$. The first are $\mathbb{S}^1$-bundles

$$\pi_m : \mathbb{S}^{2m-1} \times_{\mathbb{Z}/k} \mathbb{S}^1 \to \mathbb{S}^{2m-1}/(\mathbb{Z}/k),$$

while the second are $\mathbb{S}^{2m-1}$-bundles

$$p_m : \mathbb{S}^{2m-1} \times_{\mathbb{Z}/k} \mathbb{S}^1 \to \mathbb{S}^1/(\mathbb{Z}/k).$$

Then the line bundle associated with π_m has Euler class equal to v_m. Now, from the Gysin sequences of the sphere bundles p_m, we get the following diagram

$$\begin{array}{ccccccc} h^{q+1}(\mathbb{S}^1/(\mathbb{Z}/k)) & \xrightarrow{p^!_{m+1}} & h^{q+1}(\mathbb{S}^{2m+1} \times_{\mathbb{Z}/k} \mathbb{S}^1) & \xrightarrow{p_{m+1,!}} & h^{q-2m}(\mathbb{S}^1/(\mathbb{Z}/k)) & \xrightarrow{\cdot v_1^{m+1}} & \cdots \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle j'^!_m} & & \downarrow{\scriptstyle \cdot v_1} & & \\ h^{q+1}(\mathbb{S}^1/(\mathbb{Z}/k)) & \xrightarrow[p^!_m]{} & h^{q+1}(\mathbb{S}^{2m-1} \times_{\mathbb{Z}/k} \mathbb{S}^1) & \xrightarrow[p_{m,!}]{} & h^{q-2m+2}(\mathbb{S}^1/(\mathbb{Z}/k)) & \xrightarrow[\cdot v_1^m]{} & \cdots \end{array} \tag{8}$$

Here, since the action of $\mathbb{Z}/k$ on $\mathbb{S}^{2m-1}$ is induced from the coordinate-wise action of $\mathbb{Z}/k$ on $\mathbb{C}^m$, the vector bundle associated with p_m is just the direct sum of m line bundles associated with $p_1 = \pi_1$. Therefore, its Euler class is v_1^m and the two horizontal lines are Gysin sequences of p_m and p_{m+1}. The inclusion

$$j'_m : \mathbb{S}^{2m-1} \times_{\mathbb{Z}/k} \mathbb{S}^1 \hookrightarrow \mathbb{S}^{2m+1} \times_{\mathbb{Z}/k} \mathbb{S}^1$$

is induced by the inclusion j_m. Obviously, the left and the right squares are commutative. And the commutativity of the central square comes from the following lemma.

Lemma 4.1.2. *Let E and F be two vector bundles over X of real ranks e and f respectively. We use $f : SE \to X$ and $g : S(E \oplus F) \to X$ to denote the associated sphere bundles of E and $E \oplus F$. Then the following diagram commutes*

$$\begin{array}{ccc} h^q(S(E \oplus F)) & \xrightarrow{g_!} & h^{q-e-f+1}(X) \\ {\scriptstyle j^!}\downarrow & & \downarrow{\scriptstyle \cdot e(F)} \\ h^q(SE) & \xrightarrow[f_!]{} & h^{q-e+1}(X), \end{array}$$

where $e(F) \in h^f(X)$ is the Euler class of F.

Proof. Consider the following diagram

$$\begin{array}{ccc} SE & \xhookrightarrow{j} & S(E \oplus F) \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle g} & \downarrow{\scriptstyle p} \\ X & \xhookrightarrow[i]{} & F, \end{array}$$

where p is the natural projection onto F and i is the embedding of the zero section. One may verify easily that $i \circ g$ is homotopic to p and hence the diagram is commutative up to homotopy. Moreover, one can check without difficulty that the outer square is a Cartesian square and that Lemma 2.2.6 still holds in h^*. So we get

$$f_! j^! x = i^! p_! x = i^!(i \circ g)_! x = i^! i_! g_! x = g_! x \cdot e(F)$$

for any $x \in h^q(S(E \oplus F))$. ☐

Now we go back to the proof of Lemma 4.1.1. Consider the Gysin sequence of π_{m+1}, we have

$$h^{q+1}(\mathbb{S}^{2m+1} \times_{\mathbb{Z}/k} \mathbb{S}^1) \xrightarrow{\pi_{m+1,!}} h^q(\mathbb{S}^{2m+1}/(\mathbb{Z}/k)) \xrightarrow{\cdot v_{m+1}} h^{q+2}(\mathbb{S}^{2m+1}/(\mathbb{Z}/k)).$$

Since $x \cdot v_{m+1} = 0$, there exists $z \in h^{q+1}(\mathbb{S}^{2m+1} \times_{\mathbb{Z}/k} \mathbb{S}^1)$ such that $x = \pi_{m+1,!} z$. So we have

$$j_m^! x = j_m^! \pi_{m+1,!} z = \pi_{m,!} j_m'^! z,$$

where the last equation comes from Lemma 4.1.1 and the following Cartesian square

$$\begin{array}{ccc} \mathbb{S}^{2m-1} \times_{\mathbb{Z}/k} \mathbb{S}^1 & \xhookrightarrow{j'_m} & \mathbb{S}^{2m+1} \times_{\mathbb{Z}/k} \mathbb{S}^1 \\ {\scriptstyle \pi_m}\downarrow & & \downarrow{\scriptstyle \pi_{m+1}} \\ \mathbb{S}^{2m-1}/(\mathbb{Z}/k) & \xhookrightarrow{j_m} & \mathbb{S}^{2m+1}/(\mathbb{Z}/k). \end{array}$$

Now, we go back to the commutative diagram (8) in which z is an element of $h^{q+1}(\mathbb{S}^{2m+1} \times_{\mathbb{Z}/k} \mathbb{S}^1)$. Recall that $v_1 \in h^2(\mathbb{S}^1/(\mathbb{Z}/k)) = MU^2(X \times \mathbb{S}^1/(\mathbb{Z}/k))$ is defined by $v_1 = \gamma^! \gamma_! 1$, where $\gamma : \mathbb{S}^1/(\mathbb{Z}/k) \hookrightarrow \mathbb{S}^1 \times_{\mathbb{Z}/k} \mathbb{C}$ is the embedding of the zero section. Recall that the Gysin homomorphism $\gamma_!$ in h^*,

$$\gamma_! : h^0(\mathbb{S}^1/(\mathbb{Z}/k)) \to h^2(\mathbb{S}^1 \times_{\mathbb{Z}/k} \mathbb{C}),$$

is equal to the Gysin homomorphism $(id_X \times \gamma)_*$ in MU^*,

$$(id_X \times \gamma)_* : MU^0(X \times \mathbb{S}^1/(\mathbb{Z}/k)) \to MU^2(X \times (\mathbb{S}^1 \times_{\mathbb{Z}/k} \mathbb{C})).$$

And the pullback $\gamma^!$ is equal to $(id_X \times \gamma)^*$. Therefore, we have

$$v_1 = \gamma^! \gamma_! 1 = (id_X \times \gamma)^* (id_X \times \gamma)_* 1 = 1 \cdot \gamma^* \gamma_* 1 \in MU^2(X \times \mathbb{S}^1/(\mathbb{Z}/k)),$$

where $1 \in MU^0(X)$ is the unit and $\gamma^* \gamma_* 1 \in MU^2(\mathbb{S}^1/(\mathbb{Z}/k))$. Using geometric interpretations, we see that $MU^2(\mathbb{S}^1/(\mathbb{Z}/k)) = 0$ due to dimensional reasons, and hence we have $v_1 = 0$. Therefore, we have

$$p_{m,!} j'^!_m z = p_{m+1,!} z \cdot v_1 = 0,$$

and hence $j'^!_m z = p^!_m z'$ for some $z' \in h^{q+1}(\mathbb{S}^1/(\mathbb{Z}/k))$ from diagram (8). Denote by $i : pt = (\mathbb{Z}/k)/(\mathbb{Z}/k) \hookrightarrow \mathbb{S}^1/(\mathbb{Z}/k)$ the natural embedding and define y' to be $i^! z' \in h^{q+1}(pt)$. Then we know that $z' - y' \cdot 1 \in \widetilde{h}^{q+1}(\mathbb{S}^1/(\mathbb{Z}/k))$. By the suspension isomorphism $\widetilde{h}^{q+1}(\mathbb{S}^1/(\mathbb{Z}/k)) \cong \widetilde{h}^q(pt)$, there exists $y \in h^q(pt)$ such that $z' - y' \cdot 1 = y \cdot i_! 1$. Therefore, we have $z' = y' \cdot 1 + y \cdot i_! 1$ for some $y' \in h^{q+1}(pt)$ and $y \in h^q(pt)$. And we have

$$\begin{aligned} \pi_{m,!} j'^!_m z = \pi_{m,!} p^!_m z' &= \pi_{m,!} p^!_m (y' \cdot 1 + y \cdot i_! 1) \\ &= y' \cdot \pi_{m,!} p^!_m 1 + y \cdot \pi_{m,!} p^!_m i_! 1 \end{aligned}$$

Finally, we have $\pi_{m,!} p^!_m 1 = \pi_{m,!} 1 = \pi_{m,!} \pi^!_m 1 = 0$ by the Gysin sequence of π_m. Now we compute

$$\pi_{m,!} p^!_m i_! 1 \in h^*(\mathbb{S}^{2m-1}/(\mathbb{Z}/k)) = MU^*(X \times \mathbb{S}^{2m-1}/(\mathbb{Z}/k)).$$

Using the geometric interpretation of MU^* in Section 2, this element is represented by the product of the projection $\mathbb{S}^{2m-1} \to \mathbb{S}^{2m-1}/(\mathbb{Z}/k)$ and id_X. Therefore, it is $\phi(v_m)$ by the following lemma.

Lemma 4.1.3. *Let $f : Q \to B$ be a principal $\mathbb{Z}/k$-bundle with B compact and let $g : L = Q \times_{\mathbb{Z}/k} \mathbb{C} \to B$ be the associated line bundle. Then we have*

$$f_* 1 = \phi(e(L)) \in MU^0(B).$$

Proof. Denote by $j : Q \hookrightarrow L$ the natural embedding. From the line bundle g over B, we can construct the tautological line bundle $g^* L$ over L. We denote the zero section of L by $i : B \hookrightarrow L$ and denote the tautological section of $g^* L$

by $s : L \hookrightarrow g^*L$. Then to make L into a compact manifold, we take its one-point compactification L^* and extend g^*L to a line bundle M over it. This extension can be done since s gives a trivialization of g^*L out of $i(B)$ which is a compact subspace of L. Then we get the following diagram

$$\begin{array}{ccccccc}
Q & \overset{j}{\hookrightarrow} & L = Q \times_{\mathbb{Z}/k} \mathbb{C} & & g^*L & \hookrightarrow & M \\
{\scriptstyle f}\downarrow & & {\scriptstyle i}\uparrow\downarrow{\scriptstyle g} & & {\scriptstyle s}\uparrow\downarrow & & \downarrow \\
B & & B & \underset{g}{\longleftarrow} & L & \hookrightarrow & L^*.
\end{array}$$

Now we have

$$\phi(e(L)) = \phi(i^* i_* 1) = i^* \phi(i_* 1),$$

and hence

$$i_* \phi(e(L)) = i_* i^* \phi(i_* 1) = \phi(i_* 1) \cdot i_* 1 = [k]_F(i_* 1).$$

Since the tautological section s and the zero section intersect transversally along $i(B)$, we have $i_* 1 = e(M)$ using the geometric interpretation. Therefore, we have

$$[k]_F(i_* 1) = [k]_F(e(M)) = e(M^{\otimes k}).$$

To compute the last term, we construct a section t of the line bundle $M^{\otimes k}$ and find its vanishing locus. By definition, we have $L = \{j(q) \cdot z | q \in Q, z \in \mathbb{C}\}$ where we identifies $j(\zeta q) \cdot z$ with $j(q) \cdot \zeta z$ for any $\zeta \in \mathbb{Z}/k$. Then we define t to be

$$t(j(q) \cdot z) := (j(q) \cdot z, j(q)^{\otimes k} \cdot z^k - j(q)^{\otimes k}).$$

One can check that t can be extended to a section of M. Moreover, it is transversal to the zero section along $j(Q)$, and hence we have $j_* 1 = e(M^{\otimes k})$ as before. Meanwhile, since j is homotopic to $i \circ f$, we have $j_* 1 = i_* f_* 1$. Combining all the equations above, we get

$$i_* f_* 1 = j_* 1 = e(M^{\otimes k}) = [k]_F(i_* 1) = i_* \phi(e(L)).$$

Since i_* is an isomorphism, we have proved that $f_* 1 = \phi(e(L))$. □

Now we get back to the proof of Lemma 4.1.1. By the lemma above, we have

$$j_m^! x = \pi_{m,!} j_m'^! z = y' \cdot 0 + y \cdot \phi(v_m) = y \cdot \phi(v_m)$$

for some $y \in h^q(pt)$ and finish the proof. □

4.2 Proof of Quillen's Theorem

Equipped with the technical lemma above and Proposition 3.4.1 which relates the Steenrod operations and the Landweber–Novikov operations, we are ready to prove Quillen's Theorem now.

Recall that C is the subring of MU^* generated by the coefficients of the universal formal group law F, we can state Quillen's Theorem as follows.

Theorem 4.2.1 (Quillen's Theorem)**.** *Let X be a manifold of the homotopy type of a finite complex, then we have*

$$MU^*(X) = C \cdot \sum_{q \geqslant 0} MU^q(X),$$
$$\widetilde{MU}^*(X) = C \cdot \sum_{q > 0} MU^q(X).$$

Using the geometric interpretations, we have $MU^q(pt) = 0$ for any $q > 0$ and $MU^0(pt) \cong \mathbb{Z}$ for dimensional reasons. Thus, we get the following corollary immediately.

Corollary 4.2.2. *We have $MU^{\mathrm{ev}}(pt) = C$, and $MU^{\mathrm{odd}}(pt) = 0$.*

That is to say, we can take the coefficients of the formal group law F as a set of generators for the coefficient ring MU^*.

Before giving the proof of Quillen's Theorem, we also need the following fact from homotopy theory which does not have a proof using only geometric interpretations.

Fact 4.2.3. *If X is a manifold of the homotopy type of a finite complex, then $MU^q(X)$ is a finitely generated abelian group for any $q \in \mathbb{Z}$.*

Proof of Quillen's Theorem. By suspension isomorphisms, we only need to prove that

$$\widetilde{MU}^{\mathrm{ev}}(X) = C \cdot \sum_{q > 0} \widetilde{MU}^{2q}(X).$$

Denote the right-hand side by R. It suffices to prove that $\widetilde{MU}^{\mathrm{ev}}(X)_{(p)} = R_{(p)}$ after localization for any prime p. In the following, we will prove this equation by a decreasing induction on degrees.

To begin with, we observe that $\widetilde{MU}^{\mathrm{ev},>0}(X)_{(p)} = R^{>0}_{(p)}$ is always true since we have $1 \in C$. Now suppose that we have $\widetilde{MU}^{\mathrm{ev},>-2q}(X)_{(p)} = R^{>-2q}_{(p)}$, and we want to prove that $\widetilde{MU}^{-2q}(X)_{(p)} = R^{-2q}_{(p)}$.

To do this, we start from Proposition 3.4.1, the equation that links the Steenrod operations with the Landweber–Novikov operations. We take the principal $\mathbb{Z}/p$-bundles in the proposition to be $\mathbb{S}^{2m+1} \to \mathbb{S}^{2m+1}/(\mathbb{Z}/p)$, then we have

$$w^{n+q}P(x) = \sum_{l(\alpha) \leqslant n} w^{n-l(\alpha)} a(v)^{\alpha} s_{\alpha}(x) \in MU^{2(k-1)n-2q}(X \times \mathbb{S}^{2m+1}/(\mathbb{Z}/p))$$

for any $x \in \widetilde{MU}^{-2q}(X)$, where

$$w = w_{m+1} \in MU^{2(k-1)}(\mathbb{S}^{2m+1}/(\mathbb{Z}/p)),$$
$$v = v_{m+1} \in MU^{2}(\mathbb{S}^{2m+1}/(\mathbb{Z}/p))$$

are the Euler classes of the corresponding vector bundles on $\mathbb{S}^{2m+1}/(\mathbb{Z}/p)$, and n is sufficiently large to be valid for all m. This is possible since Proposition 3.4.1 only requires n to be large enough with respect to the dimension of X and q, and therefore has nothing to do with m.

For $\alpha \neq 0$, we have $s_\alpha(x) \in \widetilde{MU}^{\mathrm{ev},>-2q}(X)$ and hence $s_\alpha(x) \in R_{(p)}$ by the induction hypothesis. Therefore, we have

$$w^n(w^q P(x) - x) = \psi(v),$$

where $\psi(v)$ is some formal power series in v with coefficients in $R_{(p)}$. Meanwhile, by Remark 3.4.2, w can be expressed as a formal power series in v with coefficients in C and leading term $(p-1)! \cdot v^{p-1}$. Therefore we have $v^{p-1} = w \cdot \theta(v)$ for some formal power series $\theta(v)$ with coefficients in $C_{(p)}$. After multiplying both sides of the equation above by $\theta(v)^n$, we get

$$v^{n(p-1)}(w^q P(x) - x) = \psi(v) \cdot \theta(v)^n.$$

Notice that $\psi(v) \cdot \theta(v)^n$ is still a formal power series in v with coefficients in $R_{(p)}$. We may take r to be the minimal positive integer such that the equation

$$v^r(w^q P(x) - x) = \psi_1(v) \in MU^*(X \times \mathbb{S}^{2m+1}/(\mathbb{Z}/p)) \tag{9}$$

holds for some $m \in \mathbb{N}$ and for some formal power series $\psi_1(v)$ with coefficients in $R_{(p)}$.

Let $i : X \hookrightarrow \mathbb{S}^{2m+1}/(\mathbb{Z}/p) \times X$ be the map induced by the inclusion of a point into $\mathbb{S}^{2m+1}/(\mathbb{Z}/p)$. We apply $i^* : MU^*(\mathbb{S}^{2m+1}/(\mathbb{Z}/p) \times X) \to MU^*(X)$ to the equation above, then we get $\psi_1(0) = 0$ since $r \geqslant 1$ and $i^* v = 0$. Therefore, there exists a formal power series ψ_2 with coefficients in $R_{(p)}$ such that we have $\psi_1(v) = v \cdot \psi_2(v)$. Then the equation becomes

$$v \cdot [v^{r-1}(w^q P(x) - x) - \psi_2(v)] = 0.$$

Now, by the technical Lemma 4.1.1, there exists $y \in MU^{-2q+2(r-1)}(X)$ such that

$$y \cdot \phi(v) = v^{r-1}(w^q P(x) - x) - \psi_2(v),$$

or equivalently,

$$v^{r-1}(w^q P(x) - x) = \psi_2(v) + y \cdot \phi(v) \in MU^*(X \times \mathbb{S}^{2m-1}/(\mathbb{Z}/p)).$$

Here, by abuse of notation, we still use v to denote v_m instead of v_{m+1}, but one should be aware that this equation holds in $MU^*(X \times \mathbb{S}^{2m-1}/(\mathbb{Z}/p))$. Let

$$j : \mathbb{S}^{2m-1}/(\mathbb{Z}/p) \hookrightarrow \mathbb{S}^{2m-1}/(\mathbb{Z}/p) \times X$$

be the map induced by the inclusion $j' : pt \hookrightarrow X$ of a point into X. We apply

$$j^* : MU^*(\mathbb{S}^{2m-1}/(\mathbb{Z}/p) \times X) \to MU^*(\mathbb{S}^{2m-1}/(\mathbb{Z}/p))$$

to the equation above. Since all elements in R are sent to 0 due to dimensional reasons, we have $j^*(\psi_2(v)) = 0$. Meanwhile, we also have $j^*x = 0$ since x is an element in $\widetilde{MU}^{-2q}(X)$. Therefore, the equation becomes $0 = j'^*(y) \cdot \phi(v)$. Now, we replace y by $y - j'^*(y) \cdot 1$, and we may assume that $y \in \widetilde{MU}^{-2q+2(r-1)}(X)$.

If $r > 1$, we have $-2q + 2(r-1) > -2q$ and $y \in \widetilde{MU}^{\mathrm{ev},>-2q}(X)_{(p)} \subseteq R_{(p)}$ and hence $\psi_2(v) + y \cdot \phi(v)$ is still a formal power series in v with coefficients in $R_{(p)}$. Thus, $r-1$ also satisfies the equation (9) by taking $\psi_1(v)$ to be $\psi_2(v) + y \cdot \phi(v)$, which contradicts the minimality of r.

So we must have $r = 1$ and the equation becomes

$$w^q P(x) - x = \psi_2(v) + y \cdot \phi(v),$$

for some $y \in \widetilde{MU}^{-2q}(X)$.

Now, we apply i^* to the equation above and get

$$\begin{aligned} -x &= \psi_2(0) + py \quad \text{if } q > 0, \\ x^p - x &= \psi_2(0) + py \quad \text{if } q = 0. \end{aligned}$$

In the first case, since $x \in \widetilde{MU}^{-2q}(X)$ is arbitrary, we have

$$\widetilde{MU}^{-2q}_{(p)}(X) = R^{-2q}_{(p)} + p \cdot \widetilde{MU}^{-2q}_{(p)}(X).$$

We denote $N = \widetilde{MU}^{-2q}_{(p)}(X)/R^{-2q}_{(p)}$. Then N is a finitely generated $\mathbb{Z}/p$-module by Fact 4.2.3, and the equation above tells us that $N = p \cdot N$. Therefore, by Nakayama's lemma, we have $N = 0$ and hence $\widetilde{MU}^{-2q}_{(p)}(X) = R^{-2q}_{(p)}$.

In the second case, we have $x = x^p$ in $\widetilde{MU}^{0}_{(p)}(X)/(R^0_{(p)} + p\widetilde{MU}^{0}_{(p)}(X))$. Therefore, once we can prove that $\widetilde{MU}^0(X)$ is nilpotent, we have

$$\begin{aligned} \widetilde{MU}^{0}_{(p)}(X)/(R^0_{(p)} + p\widetilde{MU}^0(X)_{(p)}) = 0, \\ \textit{i.e.}, \ \widetilde{MU}^0(X)_{(p)} = R^0_{(p)} + p\widetilde{MU}^0(X)_{(p)}. \end{aligned}$$

Now, we get back to the first case, and hence by Nakayama's lemma again, we get the result.

The reason why $\widetilde{MU}^0(X)$ is nilpotent comes from the Atiyah–Hirzebruch spectral sequence. To be more concise, we have

$$E_2^{p,q} = H^p(X, MU^q) \Rightarrow MU^{p+q}(X).$$

We denote the resulting filtration of $MU^0(X)$ by

$$MU^0(X) = F^0U^0(X) \supseteq F^1U^0(X) \supseteq \ldots$$

Notice that $E_2^{0,0} = H^0(X, MU^0) = MU^0$, we have

$$F^1MU^0(X) = \ker(MU^0(X) \to MU^0) = \widetilde{MU}^0(X).$$

Since this spectral sequence is multiplicative, we have

$$\widetilde{MU}^0(X)^{\dim(X)+1} \subseteq F^{\dim(X)+1}MU^0(X) = 0.$$

The last equation is due to $E_2^{p,q} = 0$ for any $p > \dim(X)$ and any q. Therefore, we show that $\widetilde{MU}^0(X)$ is nilpotent and the proof is finished. □

4.3 The Milnor–Quillen Theorem

In this subsection, we will show how do Corollary 4.2.2 and a theorem of Lazard lead us to the structure of $MU^*(pt)$. The theorem of Lazard that we will use is proved in his paper [8]. Before stating it, let us extend our definition of the formal group law to the graded case.

Let $R = \oplus_{q\in\mathbb{Z}} R_q$ be a commutative graded ring. Then a graded formal group law over R is a formal power series $F(x, y) = \sum a_{ij}x^i y^j \in R[[x, y]]$ with the coefficients $a_{ij} \in R_{2(i+j-1)}$ that satisfies the same equations as in Definition 1.3.1. We also say F is commutative if F satisfies $F(x, y) = F(y, x)$.

The Lazard ring $\mathbb{L}$ we mentioned before has a grading that makes it into a graded commutative ring. Meanwhile, F_{univ} becomes a graded formal group law under this grading. Moreover, it turns out that F_{univ} is still the universal commutative graded formal group law. Lazard has computed $\mathbb{L}$ as a graded ring in his paper [8], which is the following theorem.

Theorem 4.3.1 (Lazard's Theorem)**.** *The Lazard ring* $\mathbb{L}$ *is a polynomial ring over* $\mathbb{Z}$ *with one generator at each positive even degree, i.e.,* $\mathbb{L} \cong \mathbb{Z}[y_1, y_2, ...]$ *with* $\deg(y_k) = 2k$ *for any* $k \in \mathbb{Z}_{>0}$.

Recall that we have defined a formal group law F on $\pi_*(MU)$ in Section 1.3. Notice that $\pi_*(MU)$ is equipped with a natural grading, it is not hard to prove that F is in fact a graded formal group law. Therefore, there exists a graded ring homomorphism $\delta : \mathbb{L} \to \pi_*(MU)$ which sends the universal formal group law F_{univ} to F. And Quillen has proved in his paper [11] that δ is in fact an isomorphism, which is the following theorem.

Theorem 4.3.2 (The Milnor–Quillen Theorem)**.** *The graded ring homomorphism* $\delta : \mathbb{L} \to \pi_*(MU)$ *which sends the universal formal group law* F_{univ} *to* F *is an isomorphism.*

Combining this theorem with Lazard's Theorem above, we are now able to prove the Milnor Theorem.

Theorem 4.3.3 (The Milnor Theorem)**.** *We have* $\pi_*(MU) \cong \mathbb{Z}[y_1, y_2, ...]$ *with the generators* $y_k \in \pi_{2k}(MU)$ *for any* $k \in \mathbb{Z}_{>0}$.

Proof of the Milnor–Quillen Theorem. Let $\epsilon : MU^*(X) \to H^*(X; \mathbb{Z})$ be the Thom homomorphism for any space X, which is defined by sending the element in $MU^*(X)$ represented by $f : Z \to X$ to f_*1 in $H^0(X, \mathbb{Z})$. Here, the map f is a proper complex-oriented map, f_*1 is the image of the unit $1 \in H^0(Z)$ under the Gysin homomorphism $f_* : H^0(Z) \to H^{-\dim(f)}(X)$. Then one can verify that ϵ commutes with pullbacks and Gysin homomorphisms. By an abuse of notation, we still use ϵ to denote the induced map

$$\epsilon : MU^*(X)[[t]] \to H^*(X)[[t]],$$

where $t = (t_1, t_2, ...)$ is a sequence of variables. Recall that we have defined the Landweber–Novikov operation $s_t : MU^*(X) \to MU^*(X)[[t]]$ in Section 3.1. Now, we define β to be

$$\beta := \epsilon \circ s_t : MU^*(X) \to MU^*(X)[[t]] \to H^*(X)[[t]].$$

Then we study the image of F_{univ} under this map.

We use e^U and e^H to denote the Euler classes in MU^* and H^* respectively, and similarly for c_t^U and c_t^H. We also use f_*^U and f_*^H to distinguish the Gysin homomorphisms in MU^* and H^*. Recall that we have

$$s_t(f_*^U x) = f_*^U(c_t^U(\nu_f) \cdot s_t x)$$

from Proposition 3.1.3. We have a similar equation for β as follows

$$\begin{aligned} \beta(f_*^U x) = \epsilon \circ s_t(f_*^U x) &= \epsilon \circ f_*^U(c_t^U(\nu_f) \cdot s_t x) \\ &= f_*^H \circ \epsilon(c_t^U(\nu_f) \cdot s_t x) \\ &= f_*^H(\epsilon(c_t^U(\nu_f)) \cdot \epsilon \circ s_t x) \\ &= f_*^H(c_t^H(\nu_f) \cdot \beta x). \end{aligned}$$

Now we are able to compute the image of Euler classes under β. Let $L \to X$ be any complex line bundle and we denote by $i : X \hookrightarrow L$ the inclusion of the zero section. Then we have

$$\beta(e^U(L)) = \beta(i^* i_*^U 1) = i^* \beta(i_*^U 1) = i^* i_*^H(c_t^H(\nu_i)),$$

where the last equation is obtained by taking $f = i$ and $x = 1$ in the equation above. Since $i : X \hookrightarrow L$ is the inclusion of the zero section, we have $\nu_i \cong L$, and hence $c_t^H(\nu_i) = c_t^H(L) = \sum_{j\geqslant 0} t_j \cdot e^H(L)^j$ with $t_0 = 1$. So we can continue the computation as follows

$$i^* i_*^H(c_t^H(\nu_i)) = i^* i_*^H(\sum_{j\geqslant 0} t_j \cdot e^H(L)^j) = \sum_{j\geqslant 0} t_j e^H(L) \cdot e^H(L)^j = \sum_{j\geqslant 0} t_j e^H(L)^{j+1}.$$

Denote by $\theta_t(T) = \sum_{j\geqslant 0} t_j T^{j+1}$ the formal power series in T with coefficients in $\mathbb{Z}[[t]]$ and $t_0 = 1$, we have proved that

$$\beta(e^U(L)) = \theta_t(e^H(L)).$$

Now, we take $L = L_1 \otimes L_2$ to be the tensor of two line bundles, then we have

$$\beta(e^U(L_1 \otimes L_2)) = \theta_t(e^H(L_1 \otimes L_2)).$$

Since $e^U(L_1 \otimes L_2) = F(e^U(L_1), e^U(L_2))$ and $e^H(L_1 \otimes L_2) = e^H(L_1) + e^H(L_2)$, we have

$$\beta(F(e^U(L_1), e^U(L_2))) = \theta_t(e^H(L_1) + e^H(L_2)). \tag{10}$$

We take $X = pt$ in β, and denote the resulting map by

$$\beta' : \pi_*(MU) = MU^*(pt) \to H^*(pt)[[t]] = \mathbb{Z}[[t]].$$

Since the coefficients of F belong to $\pi_*(MU)$, we have

$$\begin{aligned}\beta(F(e^U(L_1), e^U(L_2))) &= (\beta' F)(\beta(e^U(L_1)), \beta(e^U(L_2))) \\ &= (\beta' F)(\theta_t(e^H(L_1)), \theta_t(e^H(L_2))).\end{aligned}$$

Combining this with equation (10), we get

$$(\beta' F)(\theta_t(e^H(L_1)), \theta_t(e^H(L_2))) = \theta_t(e^H(L_1) + e^H(L_2)).$$

Since this is valid for any line bundles L_1 and L_2, in particular, we can take them to be $L_i = P_i^* \mathcal{O}(-1)$ where $P_i : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$, $i = 1, 2$, are the two projections. Then we get

$$(\beta' F)(\theta_t(T_1), \theta_t(T_2)) = \theta_t(T_1 + T_2)$$

as formal power series in T_1 and T_2. Therefore, $\theta_t(T)$ is the exponential of the formal group law $\beta' F$.

Now, we consider the composition of β' and δ, and we get

$$\begin{array}{ccccc} \mathbb{L} & \xrightarrow{\delta} & \pi_*(MU) & \xrightarrow{\beta'} & \mathbb{Z}[[t]] \\ F_{\text{univ}} & \longmapsto & F & \longmapsto & \beta' F, \end{array}$$

where the second row indicates the change of formal group laws. By Corollary 4.2.2, we know that δ is surjective. In the following, we will show that it is also injective, and hence is an isomorphism.

Now we tensor every thing with $\mathbb{Q}$, and we will show that the map $\mathbb{Q} \otimes (\beta' \delta)$ is an isomorphism by the Yoneda Lemma. Let R be any commutative $\mathbb{Q}$-algebra. Then $\mathbb{Q} \otimes (\beta' \delta)$ induces a map

$$\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[[t]], R) \to \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q} \otimes \mathbb{L}, R) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{L}, R). \tag{11}$$

By the universal property of $\mathbb{L}$, the right-hand side of (11) can be identified with the set of all formal group laws over R. Under the same identification, the left-hand side of (11) consists of formal group laws over R of the form $u\beta'\delta F$ for some map u belongs to $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[[t]], R)$. Since $\beta'\delta F$ has exponential $\theta_t(T)$, the formal group law $u\beta'\delta F$ also has exponential $\theta_u(T) = \sum_{j \geqslant 0} u(t_j) T^{j+1}$. By the formal Lie theory, which is well explained in Chapter IV of Fröhlich's book [5], every formal group law over a $\mathbb{Q}$-algebra admits a unique exponential in $R[[T]]$. Therefore, the map (11) is a bijection. And by the Yoneda Lemma, we know that $\mathbb{Q} \otimes \beta'\delta$ is an isomorphism. In particular, the morphism $\mathbb{Q} \otimes \delta$ is injective. Moreover, by Lazard's Theorem 4.3.1, $\mathbb{L}$ is a polynomial ring and hence has no torsion. Therefore, the map δ is injective and hence an isomorphism. □

Remark 4.3.4. *Historically, Milnor has computed* $\pi_*(MU)$ *and proved the Milnor Theorem in his paper [9] in 1960. Since Lazard has already proved his theorem in his paper [8] in 1955, people realized that* $\pi_*(MU)$ *and* $\mathbb{L}$ *are isomorphic as graded rings. Moreover, there is a canonical graded ring homomorphism* δ *from* $\mathbb{L}$ *to* $\pi_*(MU)$ *that sends the universal formal group law to the one over the complex cobordism ring. However, people were not certain whether* δ *gave the isomorphism between them. Finally, in 1969, Quillen showed that this was indeed the case in his paper [11], and this result is now called the Milnor–Quillen Theorem.*

References

[1] J.F. Adams. "On the structure and applications of the steenrod algebra". In: *Commentarii Mathematici Helvetici* 32.1 (Dec. 1958), pp. 180–214.

[2] J.F. Adams. *Stable Homotopy and Generalised Homology*. Chicago Lectures in Mathematics. University of Chicago Press, 1974. ISBN: 9780226005249. URL: https://books.google.fr/books?id=6vG13YQcPnYC.

[3] M. Audin. *Torus Actions on Symplectic Manifolds*. Progress in Mathematics. Birkhäuser Basel, 2012. ISBN: 9783034879606. URL: https://books.google.fr/books?id=Qh_yBwAAQBAJ.

[4] Edgar H. Brown. "Cohomology Theories". In: *Annals of Mathematics* 75.3 (1962), pp. 467–484. ISSN: 0003486X. URL: http://www.jstor.org/stable/1970209 (visited on 09/22/2023).

[5] A. Frohlich. *Formal Groups*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1968. ISBN: 9783540042440. URL: https://books.google.fr/books?id=_TwZAQAAIAAJ.

[6] Stanley O. Kochman. "Bordism, Stable Homotopy and Adams Spectral Sequences". In: 1996.

[7] A. Kōno and D. Tamaki. *Generalized Cohomology*. Generalized Cohomology Volume 230. American Mathematical Society, 2006. ISBN: 9780821835142. URL: https://books.google.fr/books?id=3HY4ruJ6BigC.

[8] Michel Lazard. "Sur les groupes de Lie formels à un paramètre". fre. In: *Bulletin de la Société Mathématique de France* 83 (1955), pp. 251–274. URL: http://eudml.org/doc/86893.

[9] John W. Milnor. "On the Cobordism Ring Ω^* and a Complex Analogue, Part I". In: *American Journal of Mathematics* 82 (1960), p. 505. URL: https://api.semanticscholar.org/CorpusID:124508043.

[10] Daniel Quillen. "Elementary proofs of some results of cobordism theory using Steenrod operations". In: *Advances in Mathematics* 7.1 (1971), pp. 29–56. ISSN: 0001-8708. DOI: https://doi.org/10.1016/0001-8708(71)90041-7. URL: https://www.sciencedirect.com/science/article/pii/0001870871900417.

[11] Daniel Quillen. "On the formal group laws of unoriented and complex cobordism theory". In: (1969).

[12] D.C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres*. ISSN. Elsevier Science, 1986. ISBN: 9780080874401. URL: https://books.google.fr/books?id=gnTDf9wpB9sC.

[13] Réné Thom. "Quelques propriétés globales des variétés différentiables." In: *Commentarii mathematici Helvetici* 28 (1954), pp. 17–86. URL: http://eudml.org/doc/139072.